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Singular pressures in the mechanics of compressible fluids

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CHAPTER 1

INTRODUCTION

In fluid mechanics, the Navier Stokes system for a compressible fluid is a system of nonlinear partial differential equations that describe the motion of a fluid through the study of its velocity field, its density and its internal energy. This system was introduced after the observations of Euler in the middle of the 18th century and of Navier in 1827, Poisson in 1829 and Stokes in 1845 [24]. Until now, a complete theory on the solvability and stability, mainly in two and three space dimension, has not yet be found despite being the focus of many researchers such as Solonnikov, P.L Lions, E. Feireisl, Matsumura, Nishida, etc.

In one dimensional space, the qualitative properties of this system are better understood. For instance, when the viscosity is constant, existence and uniqueness of classical solution associated to the Navier Stokes system with regular initial data was obtained in the 60's thanks to the works of Kanel [15], Kazhikhov [16] and the Russian school [1]. Likewise, in [10, 11, 14], David Hoff proved the existence of global weak solutions with initial density admitting shocks (discontinuities), we refer also to the works of Serre [23, 22]. When the viscosity is not constant, existence and uniqueness of classical solution for the Cauchy problem associated to the Navier Stokes with regular initial data was obtained by Mellet and Vasseur [19], Haspot [9], Constantin and al [6] and Burtea, Haspot [4], whereas weak solutions with discontinuous density are recently obtained by Burtea, Haspot [5]. In their recent work [2], Bresch, Perrin and Zatorska justified mathematically the derivation of the viscous free/congested zones two-phase model from the Navier Stokes system for a compressible isentropic fluid with a singular pressure. In order to handle it, the authors study the Navier Stokes system in one space dimension with singular pressure and with initial data belonging to $H^1(0,1)$, in particular continuous. Motivated by this work, we want to know if regularity on the initial data can be reduced. Our goal is to seek if it is possible to obtain a weak solution of the Navier Stokes system assuming a Lebesgue conditions on initial date. In particular, the initial density may be discontinuous. The remainder of this document is structured as follows:

- in the first chapter we begin by recalling the physical principles being behind the Navier Stokes system for compressible fluid at constant temperature for the one dimensional problem. In a second time, we will introduce the mass-Lagrangian formulation.

- In the second one, we study the local theory for regular initial data of the Navier Stokes system with singular pressure. In fact, having proved the existence and uniqueness of local strong solution, we deal with the long time existence of the local solution.

- In the third chapter, one proves the global well-posedness of classical solution of the Navier Stokes system with a particular singular pressure law. One supposes that the initial density is away from zero and the singularity point of the pressure.

- In the last chapter, we construct a weak solution of the Navier Stokes system with singular pressure by assuming that the initial energy is small and the initial density is far from vacuum and the singularity point of the pressure. The classical energy estimate leads to the fact that the weak velocity u is just $L^2_{\text{loc}}(\mathbb{R}^+, H^1(\mathbb{T}^1))$ which does not allow us to define formally the flow associated to the velocity u. Using techniques introduced by Hoff in [11, 12], we recover that $\partial_x u \in L^1_{\text{loc}}(\mathbb{R}^+, L^\infty(\mathbb{T}^1))$. That is sufficient to justify the global well-posedness of the flow associated to the velocity u and helps to understand more the transport equation in the case of discontinuous initial density.

CHAPTER 2

PRESENTATION OF THE MODEL

In this chapter, we first establish the Navier Stokes system from the point of view of physics. In a second time, we will discuss the mass-Lagrangian change of variable, which is a feature of the Navier Stokes system in one space dimensional.

2.1 Navier Stokes system

The motion of a fluid at constant temperature is described by a system of two evolutionary equations called the Navier Stokes system : the first equation is called the mass conservation equation. It comes from the fact that in any evolving volume, the mass is conserved : the fluid is advected. The second one, called momentum equation is a simple application of the Newton's second law. In what follows we recall the physical considerations that lead to the two equations.

We consider a fluid of density ρ and velocity u in motion with a constant temperature and we introduce the flow of u :

$$\mathcal{X}_t(x) = x + \int_0^t u(s, \mathcal{X}_s(x)ds.$$

Let V be a fluid element driven by the flow $\mathcal{X}_t$. Mass of fluid in $\mathcal{X}_t(V)$ is :

$$m(\mathcal{X}_t(V)) = \int_{\mathcal{X}_t(V)} \rho(t,x)dx.$$

As V is advected with the velocity u, the quantity of mass of fluid contained in the material volume $\mathcal{X}_t(V)$ being constant over time, one has :

$$\frac{d}{dt} m(\mathcal{X}_t(V)) = 0.$$

Then applying the Liouville transport equation Theorem A.1.1, one has :

$$\frac{d}{dt} m(\mathcal{X}_t(V)) = \int_{\mathcal{X}_t(V)} \left(\partial_t \rho + \mathrm{div}\,(\rho u)\right)(t,x)dx = 0.$$

This relation being true for any element of volume V then, one obtains the mass conservation equation

$$\partial_t \rho + \mathrm{div}\,(\rho u) = 0. \tag{2.1}$$

Now, we apply Newton's second law which reads as follows : *the rate of change of the total momentum of an element of fluid occupying a domain $\mathcal{X}_t(V)$ at each time is equal to force acting on $\mathcal{X}_t(V)$.* To apply this law, one must take into account of all the forces acting on $\mathcal{X}_t(V)$. There are of two types : *stress forces* and *external or body forces.* As the name suggests, contact forces characterise the contact interaction between material elements. They depend on the rheology of the fluid. If σ is the stress tensor, $\vec{n}$ the unit normal vector field on $\partial\mathcal{X}_t(V)$ and ds the surface element of $\partial\mathcal{X}_t(V)$, contact forces acting on $\mathcal{X}_t(V)$ is given by :

$$\int_{\partial\mathcal{X}_t(V)} \sigma(t,x)\cdot\vec{n}(t,x)ds = \int_{\mathcal{X}_t(V)} \mathrm{div}\,(\sigma)(t,x)dx.$$

Exterior forces are those acted by external system. They can be gravity, friction, etc. If f is the density of external forces, then external forces acting on $\mathcal{X}_t(V)$ is given by :

$$\int_{\mathcal{X}_t(V)} f(t,x)dx.$$

Noting that quantity of momentum in a volume $\mathcal{X}_t(V)$ of fluid is

$$\int_{\mathcal{X}_t(V)} \rho(t,x)u(t,x)dx$$

and applying Newton's second law and performing the transport theorem of Liouville Corollary A.1.1, one obtains :

$$\int_{\mathcal{X}_t(V)} \left(\partial_t(\rho u) + \mathrm{div}\,(\rho u\otimes u)\right)(t,x)dx = \int_{\mathcal{X}_t(V)} \mathrm{div}\,(\sigma)(t,x)dx + \int_{\mathcal{X}_t(V)} f(t,x)dx.$$

As this is true for any V then, one obtains the second equation of Navier Stokes system referred as momentum equation :

$$\partial_t(\rho u) + \mathrm{div}\,(\rho u\otimes u) = \mathrm{div}\,(\sigma) + f. \tag{2.2}$$

Gathering (2.1) and (2.2), one concludes that the motion of a compressible fluid with constant temperature is described by the Navier Stokes system :

$$\begin{cases} \partial_t\rho + \mathrm{div}\,(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u\otimes u) = \mathrm{div}\,(\sigma) + f. \end{cases}$$

When the temperature is not constant, one has an additional equation for the temperature or the energy, to describe fully the motion of the fluid.

A fluid is said to be *Newtonian* if its stress tensor is:

$$\sigma = -pI_d + \lambda\,\mathrm{div}\,(u)I_d + 2\mu Du$$

where p is the *pressure* of the fluid, λ and μ are called Lamé coefficients and where

$$Du = \frac{1}{2}\left(\nabla u +^t \nabla u\right)$$

is the deformation tensor. In general, λ and μ are functions depending on the density and we say that the fluid is *barotropic*. If they are constant for a fluid, we say that such fluid is *isentropic*. We conclude that for *Newtonian, barotropic, isothermal* fluid, the Navier Stokes system is written :

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P = \nabla(\lambda \operatorname{div}(u)) + \mu \Delta u + f. \end{cases} \tag{2.3}$$

The pressure is a thermodynamic variable that models the interaction between fluid particles. Particular examples are the following :

1. perfect Gas pressure law : For perfect gas, the pressure is the form $P(\rho) = a\rho$;
2. adiabatic pressure law : $P(\rho) = a\rho^\gamma$ with $\gamma > 1$;
3. Van-der-Walls pressure law : $P(\rho) = \dfrac{a\rho}{1 - b\rho} + c\rho^2$.

2.2 Change of variables

In this section, we exploit the structure of the one dimensional Navier Stokes system to obtain it in new variable, see more details in [1]. We consider the Navier Stokes system for Newtonian isothermal and barotropic fluid on the one dimensional torus $\mathbb{T}^1$, so with periodic boundary conditions :

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u \otimes u) + \partial_x P = \partial_x(\mu \partial_x u). \end{cases} \tag{2.4}$$

Let ρ_0 be the initial density satisfying $\inf\limits_{x \in \mathbb{T}^1} \rho_0(x) = \alpha_0 > 0$. Let us consider the mapping $x \mapsto \dfrac{1}{M} \displaystyle\int_0^x \rho_0(s) ds$ with $M = \displaystyle\int_{\mathbb{T}^1} \rho_0(x) dx$. This map is one-to-one on $\mathbb{T}^1$ to itself and its inverse denoted Y satisfies

$$Y'(m) = \frac{1}{(Y^{-1})'(Y(m))} = \frac{M}{\rho_0(Y(m))}.$$

Suppose that the velocity $u \in L^1_{\text{loc}}(\mathbb{R}^+, H^2(\mathbb{T}^1))$, one can define the flow $X(\cdot, x) : \mathbb{R}^+ \to \mathbb{T}^1$

$$\begin{cases} \dfrac{dX}{dt}(t,x) = u(t, X(t,x)), \\ X(0,x) = x. \end{cases}$$

X is given by the Duhamel formula

$$X(t,x) = x + \int_0^t u(s, X(s,x) ds.$$

Notation. *The mass Lagrangian change of variable is $J\colon (t,m) \mapsto (t, X(t, Y(m))$. For any function $g\colon (t,x) \mapsto g(t,x)$, we denote $\widetilde{g}$ the function defined by :*

$$\widetilde{g}(t,m) = g(t, X(t, Y(m)).$$

Proposition 2.2.1. *For any function g, one has :*

$$\begin{cases} \partial_m \tilde{g}(t,m) = \dfrac{M}{\rho_0((Y(m))}\dfrac{\partial X}{\partial x}(t,Y(m))\widetilde{\partial_x g}(t,m), \\ \partial_t \tilde{g}(t,m) = \widetilde{\dfrac{\partial g}{\partial t}}(t,m) + \tilde{u}(t,m)\widetilde{\partial_x g}(t,m), \\ \dfrac{\partial x}{\partial m} = \dfrac{M}{\rho_0(Y(m))}\dfrac{\partial X}{\partial x}\,(t,Y(m))\,. \end{cases}$$

Remark 2.2.1. *The new coordinate m is referred as mass-Lagrangian coordinate whereas x is called Eulerian coordinate. We will write (2.4) into this new coordinate. We begin by the mass equation $(2.4)_1$.*

Using the previous proposition, one has :

$$\partial_t\widetilde{\rho}(t,m) = \widetilde{\partial_t\rho}(t,m) + \widetilde{u}(t,m)\widetilde{\partial_x\rho}(t,m)$$

but by $(2.4)_1$,

$$-\widetilde{\partial_t\rho}(t,m) = \widetilde{\partial_x(\rho u)} = \widetilde{u}(t,m)\widetilde{\partial_x\rho}(t,m) + \widetilde{\rho}(t,m)\widetilde{\partial_x u}(t,m)$$

then

$$\partial_t\widetilde{\rho}(t,m) = -\widetilde{\rho}(t,m)\widetilde{\partial_x u}(t,m). \tag{2.5}$$

Furthermore,

$$\frac{d}{dt}\left(\rho(t,X(t,x))\frac{\partial X}{\partial x}(t,x)\right) = 0$$

so

$$\frac{1}{M}\rho_0\left(Y(m)\right)\left(\frac{\partial X}{\partial x}(t,Y(m))\right)^{-1} = \widetilde{\rho}(t,m)$$

and using the previous proposition to express $\widetilde{\partial_x u}(t,m)$ in (2.5), one has

$$\partial_t\widetilde{\rho} = -\widetilde{\rho}^2\partial_m\widetilde{u}. \tag{2.6}$$

Moreover, using the third identity of the previous proposition, one notes that

$$\frac{\partial x}{\partial m} = \frac{1}{\widetilde{\rho}(t,m)}. \tag{2.7}$$

So, in mass-Lagrangian coordinate $(2.4)_1$ is equivalent to the equation (2.6). We turn now to the momentum equation in mass-Lagrangian coordinate.

Testing $(2.4)_2$ with a smooth function $\psi \in \mathscr{C}_0((0,+\infty)\times\mathbb{T}^1)$, one has :

$$\int_0^T\int_{\mathbb{T}^1}\left[\rho u\partial_t\psi + \left(\rho u^2 + P(\rho) - \mu(\rho)\partial_x u\right)\partial_x\psi\right]dtdx = 0.$$

For $\varphi \in \mathcal{D}((0,T)\times\mathbb{T}^1)$, one can use $\psi = \varphi\circ J^{-1}$ as test function in the above formulation. By doing like that, and next performing the change of variable J, one obtains :

$$\int_0^T\int_{\mathbb{T}^1}\widetilde{u}\left[\partial_t(\widetilde{\varphi\circ X^{-1}}) + \widetilde{u}(t,m)\partial_x(\widetilde{\varphi\circ X^{-1}})\right] + \left[P(\widetilde{\rho}) - \mu(\widetilde{\rho})\widetilde{\partial_x u}\right]\partial_x(\widetilde{\varphi\circ X^{-1}})\times\frac{1}{\widetilde{\rho}(t,m)}dtdm = 0.$$

Next, using Proposition 2.2.1, one obtains :

$$\int_0^T \int_{\mathbb{T}^1} \left[\partial_t \widetilde{u} + \partial_m \left(P(\widetilde{\rho}) - \mu(\widetilde{\rho})\widetilde{\rho}\partial_m \widetilde{u}\right)\right] \varphi dt dm = 0, \forall \varphi \in \mathcal{D}((0,T)\times\mathbb{T}^1)$$

then, the momentum equation in mass-Lagrangian coordinate is :

$$\partial_t \widetilde{u} + \partial_m \left[P(\widetilde{\rho}) - \mu(\widetilde{\rho})\widetilde{\rho}\partial_m \widetilde{u}\right] = 0. \tag{2.8}$$

Gathering (2.6) and (2.8), one notes that the Navier Stokes system in mass-Lagrangian coordinate is :

$$\begin{cases} \partial_t \widetilde{\rho} + \widetilde{\rho}^2 \partial_m \widetilde{u} = 0, \\ \partial_t \widetilde{u} + \partial_m \left(P(\widetilde{\rho}) - \mu(\widetilde{\rho})\widetilde{\rho}\partial_m \widetilde{u}\right) = 0. \end{cases} \tag{2.9}$$

Thus setting $\tau(t,m) = \dfrac{1}{\widetilde{\rho}(t,m)}$ called *specific volume* and $v(t,m) = \widetilde{u}(t,m)$ we have :

$$\begin{cases} \partial_t \tau - \partial_m v = 0, \\ \partial_t v + \partial_m \left(P(1/\tau) - \mu(1/\tau)\tau^{-1}\partial_m v\right) = 0. \end{cases}$$

Remark 2.2.2. *This change of variable is only available in one dimension. In higher dimension, one is satisfied just with the change in Lagrangian coordinate X, for more details, refer to [7].*

CHAPTER 3 LOCAL THEORY FOR REGULAR INITIAL DATA

In this chapter we develop a local theory of the Navier Stokes system with singular pressure. It is divided into four sections. In the first one, we use a fixed point argument to prove the existence of local solution of the Navier Stokes system in mass-Lagrangian coordinate, and due to the regularity of the flow, we deduce the existence of classical solution of the system in *Eulerian* coordinate. In the second, we prove the uniqueness of the local solution by proving a ≪ *continuity* ≫ of the solution with respect to initial data. In the last section, we prove a blow-up criteria similar to the one in [4] that to say L^∞-norm of the density and its inverse control the higher Sobolev norms of solutions provided that it is away from the singularity of the pressure.

3.1 Existence of strong solution

We consider the Navier Stokes system :

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + P(\rho) - \mu(\rho)\partial_x u) = 0. \end{cases} \tag{3.1}$$

Above, the viscosity $\mu \in \mathscr{C}^3([0, +\infty))$ is a positive function of ρ and the pressure $P\colon [0, \rho_{max}) \mapsto \mathbb{R}^+ \in \mathscr{C}^3([0, \rho_{max}))$, increasing function of ρ satisfies :

$$\lim_{s \to \rho_{max}} P(s) = +\infty. \tag{3.2}$$

The main result of this section is summarised in the following theorem.

Theorem 3.1.1. *Assume that $(\rho_0, u_0) \in H^2(\mathbb{T}^1) \times H^2(\mathbb{T}^1)$, the viscosity μ, the pressure P are like above and that there exists $\alpha_0 > 0$ such that*

$$\alpha_0 \leqslant \rho_0 \leqslant \rho_{max} - \alpha_0.$$

Then, there exists $T > 0$, a classical solution

$$(\rho, u) \in \mathscr{C}([0, T], H^2(\mathbb{T}^1)) \times \big(\mathscr{C}([0, T], H^2(\mathbb{T}^1)) \cap L^2((0, T), H^3(\mathbb{T}^1))\big)$$

of the Cauchy problem associated to the Navier Stokes system (3.1) with initial data (ρ_0, u_0).

Moreover, there exist two constants $\eta > 0$ and $C > 0$ depending on T, α and on the initial data such that :

$$\eta \leqslant \rho \leqslant \rho_{max} - \eta,$$

$$\sup_{0\leqslant t\leqslant T} \{\|\rho(t)\|^2_{H^2(\mathbb{T}^1)} + \|u(t)\|^2_{H^2(\mathbb{T}^1)}\} + \int_0^T |\partial_x u|^2_{H^2(\mathbb{T}^1)}\} \leqslant C.$$

In order to prove Theorem 3.1.1, we will work in mass-Lagrangian coordinate : first, we solve (3.3) with a fixed point argument and next we pass to the *Eulerian* coordinate. Let us note that, for a priori estimates, it is easier to manipulate the Navier Stokes system in mass-Lagrangian rather than in *Eulerian* coordinate. For instance, this change of variable makes the mass conservation equation very easy to solve. To be convinced, we refer to [20], where John Nash showed the existence of classical solution with Hölder regularity for the Navier Stokes system while working in Eulerian coordinates. We also refer to [7] where, R. Danchin proved existence of solution with Besov regularity for the Navier Stokes system while working in Lagrangian coordinates.

We complete the above existence theorem by an extension one, which is stated as follows :

Theorem 3.1.2. *Assume that $(\rho_0, u_0) \in H^2(\mathbb{T}^1) \times H^2(\mathbb{T}^1)$, the viscosity $\mu(\rho) = \rho^\theta$, $\theta \geqslant 0$, the pressure P is like in (3.2) and that there exists $\alpha_0 > 0$ such that*

$$\alpha_0 \leqslant \rho_0 \leqslant \rho_{max} - \alpha_0.$$

Let $(\rho, u) \in \left(\mathscr{C}([0,T], H^2(\mathbb{T}^1))\right)^2$ a local solution of the Cauchy problem associated to (3.1) and initial data (ρ_0, u_0). If we further assume that there exists $C = C(T) > 0$ such that

$$\forall\, 0 \leqslant t < T, \quad C(T) \leqslant \rho \leqslant \rho_{max} - C(T)$$

then the classical solution (ρ, u) can be extended beyond T.

The proof of this theorem is done in section 3.3 of this chapter. It is a direct consequence of the blow-up criterion of Navier Stokes system in mass-Lagrangian coordinate Theorem 3.3.2.

In consequence, one obtains the following blow-up criterion.

Theorem 3.1.3. *Assume that $(\rho_0, u_0) \in H^2(\mathbb{T}^1) \times H^2(\mathbb{T}^1)$, the viscosity $\mu(\rho) = \rho^\theta$, $\theta \geqslant 0$, the pressure P is like in (3.2) and that there exists $\alpha_0 > 0$ such that*

$$\alpha_0 \leqslant \rho_0 \leqslant \rho_{max} - \alpha_0.$$

Let (ρ, u) a local solution of the Cauchy problem associated to (3.1) and initial data (ρ_0, u_0). Let T^* be the maximal existence time of the solution. If we further assume that

$$0 < \inf_{[0,T^*)\times\mathbb{T}^1} \rho \leqslant \sup_{[0,T^*)\times\mathbb{T}^1} \rho < \rho_{max}$$

then $T^* = +\infty$.

This theorem can be easily deduced from Theorem 3.1.2 by absurd so, for the sake of brevity it will not be proved.

As we know, in mass-Lagrangian coordinate (3.1) is

$$\begin{cases} \partial_t \tau - \partial_m v = 0, \\ \partial_t v + \partial_m \Big(\widetilde{P}(\tau) - \widetilde{\mu}(\tau)\partial_m v \Big) = 0. \end{cases} \tag{3.3}$$

Above, functions $\widetilde{P}$ and $\widetilde{\mu}$ are defined by :

$$\widetilde{P}(\tau) = P(1/\tau) \quad \text{and} \quad \widetilde{\mu}(\tau) = \tau^{-1}\mu(1/\tau).$$

Notation. *The linearized problem associated to (3.3) is :*

$$\begin{cases} \partial_t v - \partial_m(V\partial_m v) = F, \\ v(0) = v_0 \end{cases}$$

with some functions V and F.

In the following subsection we study the linearized problem associated to (3.3) in order to set up tools for fixed point theorem in the second subsection.

3.1.1 Study of the linearized problem

Let us consider the linear non-homogeneous parabolic equation

$$\partial_t v - \partial_m(V\partial_m v) = F \tag{3.4}$$

with V and F two given functions, periodic in space, satisfying :

$$\begin{cases} V(0) \in L^\infty(\mathbb{T}^1), \quad \partial_t V \in L^1_{\text{loc}}(\mathbb{R}^+, L^\infty(\mathbb{T}^1)), \\ \partial_m V \in L^4_{\text{loc}}(\mathbb{R}^+, L^2(\mathbb{T}^1)), \quad \inf V = \underline{V} > 0, \\ F \in L^2_{\text{loc}}(\mathbb{R}^+, L^2(\mathbb{T}^1)). \end{cases} \tag{3.5}$$

We aim to prove that the Cauchy problem associated to (3.4) with initial data $v_0 \in H^2(\mathbb{T}^1)$ has a unique solution with the same regularity as in Theorem 3.1.1, that is to say $\mathscr{C}(\mathbb{R}^+, H^2(\mathbb{T}^1)) \cap L^2_{\text{loc}}(\mathbb{R}^+, H^3(\mathbb{T}^1))$. We begin by the following existence and uniqueness theorem :

Theorem 3.1.4. *Assume $v_0 \in H^1(\mathbb{T}^1)$ and V, F satisfy (3.5). Then, there exists a unique $v \in \mathscr{C}(\mathbb{R}^+, H^1(\mathbb{T}^1)) \cap L^2_{loc}(\mathbb{R}^+, H^2(\mathbb{T}^1))$ such that $\partial_t v \in L^2_{loc}(\mathbb{R}^+, L^2(\mathbb{T}^1))$, solution of the Cauchy problem*

$$\begin{cases} \partial_t v - \partial_m(V\partial_m v) = F, \\ v(0) = v_0. \end{cases} \tag{3.6}$$

Moreover, for all $T > 0$ there is a constant $C = C(T, V) > 0$ such that :

$$\|v\|^2_{\mathscr{C}([0,T],H^1(\mathbb{T}^1))} + \|\partial_t v\|^2_{L^2((0,T)\times\mathbb{T}^1)} + \underline{V}\|\partial_m v\|^2_{L^2((0,T),H^1(\mathbb{T}^1))} \leqslant C\left(\|v_0\|^2_{H^1(\mathbb{T}^1)} + \|F\|^2_{L^2((0,T)\times\mathbb{T}^1)}\right). \tag{3.7}$$

The constant C is given by

$$C = C_1 \exp\left(T + C_1\|\partial_m V\|^4_{L^4((0,T),L^2(\mathbb{T}^1))}\right) + \max\left(1, \|V(0)\|_{L^\infty(\mathbb{T}^1)}\right) \exp\left(\|\partial_t V\|_{L^1((0,T),L^\infty(\mathbb{T}^1))}\right) \tag{3.8}$$

where C_1 depends only on $\underline{V}$.

It turns out that if we have more information on V, F and the initial data v_0, the solution will be more regular. This result is stated in the following.

Theorem 3.1.5. *Assume that functions V and F satisfy (3.5) and also*

$$\partial_m V \in L^\infty_{loc}(\mathbb{R}^+, L^\infty(\mathbb{T}^1)), \quad \partial_{mm} V \in L^\infty_{loc}(\mathbb{R}^+, L^2(\mathbb{T}^1)).$$

If $v_0 \in H^2(\mathbb{T}^1)$ and $F \in L^2_{loc}(\mathbb{R}^+, H^1(\mathbb{T}^1))$ then there exists a unique $v \in \mathscr{C}(\mathbb{R}^+, H^2(\mathbb{T}^1)) \cap L^2_{loc}(\mathbb{R}^+, H^3(\mathbb{T}^1))$ such that $\partial_t v \in L^2_{loc}(\mathbb{R}^+, H^1(\mathbb{T}^1))$, solution of the Cauchy problem associated to (3.6) and v_0. Moreover for all $T > 0$, there exists $C = C(T, V) > 0$ such that :

$$\begin{aligned}\|v\|^2_{\mathscr{C}([0,T],H^2(\mathbb{T}^1))} + \|\partial_t v\|^2_{L^2((0,T),H^1(\mathbb{T}^1))} + \underline{V}\|\partial_m v\|^2_{L^2((0,T)\times H^2(\mathbb{T}^1))} \\ \leqslant C\left(\|v_0\|^2_{H^2(\mathbb{T}^1)} + \|F\|^2_{L^2((0,T),H^1(\mathbb{T}^1))}\right). \end{aligned} \tag{3.9}$$

We will prove Theorem 3.1.4, with a homotopy argument, we define, for $T > 0$, two sets X, Y and function V_θ by :

$$X = \{v \in \mathscr{C}([0,T], H^1(\mathbb{T}^1)) \cap L^2((0,T), H^2(\mathbb{T}^1))\colon \partial_t v \in L^2((0,T) \times \mathbb{T}^1)\},$$

$$Y = L^2((0,T) \times (\mathbb{T}^1)) \times H^1(\mathbb{T}^1) \quad \text{and} \quad V_\theta = (1-\theta)\underline{V} + \theta V \quad \theta \in [0,1]$$

and consider the map :

$$\begin{aligned}\Pi_\theta\colon X &\to Y \\ v &\mapsto (\partial_t v - \partial_m(V_\theta \partial_m v); v(0)).\end{aligned} \tag{3.10}$$

We will show that the set

$$\mathscr{E} = \{\theta \in [0,1] : \Pi_\theta \text{ is one-to-one}\} \tag{3.11}$$

is at the same time open, closed and nonempty set of $[0,1]$, thus $1 \in \mathscr{E}$.

Proof of Theorem 3.1.4. Let us begin the proof of Theorem 3.1.4 by a uniqueness property for equation (3.4).

Theorem 3.1.6. *Assume that $F \in L^2((0,T), H^{-1}(\mathbb{T}^1))$, $v_0 \in L^2(\mathbb{T}^1)$ and V satisfying (3.5). Then there exists, in*

$$X = \{v \in \mathscr{C}([0,T], L^2(\mathbb{T}^1)) \cap L^2((0,T), H^1(\mathbb{T}^1))\colon \partial_t v \in L^2((0,T), H^{-1}(\mathbb{T}^1))\}$$

at most one solution of the Cauchy problem

$$\begin{cases}\partial_t v - \partial_m (V\partial_m v) = F, \\ v(0) = v_0.\end{cases} \tag{3.12}$$

Proof. Let v_1 and v_2 be two solutions of the above equation in X. Obviously, $w := v_2 - v_1$ is solution of the homogeneous equation

$$\begin{cases}\partial_t w - \partial_m (V\partial_m w) = 0, \\ v(0) = 0.\end{cases}$$

As $\partial_t V \in L^1((0,T), L^\infty(\mathbb{T}^1))$ then $V \in \mathscr{C}([0,T], L^\infty(\mathbb{T}^1))$ and so $\partial_m(V\partial_m v) \in L^2((0,T), H^{-1}(\mathbb{T}^1))$. We are able to multiply the first line of the above equation by v, and obtain after integrating on $\mathbb{T}^1$,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^1}|w|^2 + \underline{V}\int_{\mathbb{T}^1}|\partial_m w|^2 = 0.$$

Next, integrating the above equation on time, one obtains :

$$\|w\|^2_{\mathscr{C}([0,T],L^2(\mathbb{T}^1))} + \underline{\mathrm{V}}\int_0^T\int_{\mathbb{T}^1}|\partial_m w|^2 = 0.$$

This achieves the proof. □

Next we state an a priori estimates :

Lemma 3.1.1. *Suppose that $v_0 \in H^1(\mathbb{T}^1)$ and $v \in \mathscr{C}([0,T], H^1(\mathbb{T}^1)) \cap L^2((0,T), H^2(\mathbb{T}^1))$ and $\partial_t v \in L^2((0,T)\times\mathbb{T}^1)$ satisfy (3.6). Then there exists a constant $C = C(T,V) > 0$ such that v satisfies the estimate (3.7).*

Proof. Multiplying (3.6) by v, integrating over $\mathbb{T}^1$ with respect to m, one obtains

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2((\mathbb{T}^1))} + \underline{\mathrm{V}}\|\partial_m v(t)\|^2_{L^2((\mathbb{T}^1))} \leqslant \frac{1}{2}\|F(t)\|^2_{L^2((\mathbb{T}^1))} + \frac{1}{2}\|v(t)\|^2_{L^2((\mathbb{T}^1))}. \tag{3.13}$$

Next, multiplying (3.6) by $\partial_{mm}v$ integrating over $\mathbb{T}^1$ with respect to m, one obtains after some integration by part,

$$\frac{1}{2}\frac{d}{dt}\|\partial_m v\|^2_{L^2(\mathbb{T}^1)} + \int_{\mathbb{T}^1} V|\partial_{mm}v|^2 = \int_{\mathbb{T}^1} F\partial_{mm}v - \int_{\mathbb{T}^1}\partial_{mm}v\partial_m v\partial_m V. \tag{3.14}$$

By Holder inequality, one obtains :

$$\left|\int_{\mathbb{T}^1}\partial_{mm}v\partial_m v\partial_m V\right| \leqslant \|\partial_{mm}v\|_{L^2(\mathbb{T}^1)}\|\partial_m v\|_{L^\infty(\mathbb{T}^1)}\|\partial_m V\|_{L^2(\mathbb{T}^1)}$$

and by Gagliardo-Niremberg inequality

$$\|\partial_m v\|^2_{L^\infty(\mathbb{T}^1)} \leqslant 2\|\partial_{mm}v\|_{L^2(\mathbb{T}^1)}\|\partial_m v\|_{L^2(\mathbb{T}^1)},$$

then

$$\left|\int_{\mathbb{T}^1}\partial_{mm}v\partial_m v\partial_m V\right| \leqslant \sqrt{2}\|\partial_{mm}v\|^{3/2}_{L^2(\mathbb{T}^1)}\|\partial_m v\|^{1/2}_{L^2(\mathbb{T}^1)}\|\partial_m V\|_{L^2(\mathbb{T}^1)}.$$

Using Young inequality for real numbers, for $\varepsilon > 0$, there is $C_\varepsilon > 0$ such that

$$\left|\int_{\mathbb{T}^1}\partial_{mm}v\partial_m v\partial_m V\right| \leqslant \varepsilon\|\partial_{mm}v\|^2_{L^2(\mathbb{T}^1)} + C_\varepsilon\|\partial_m V\|^4_{L^2(\mathbb{T}^1)}\|\partial_m v\|^2_{L^2(\mathbb{T}^1)}. \tag{3.15}$$

As well, by Hölder inequality, one has :

$$\left|\int_{\mathbb{T}^1}\partial_{mm}vF\right| \leqslant \|\partial_{mm}v\|_{L^2(\mathbb{T}^1)}\|F\|_{L^2(\mathbb{T}^1)} \leqslant \varepsilon\|\partial_{mm}v\|^2_{L^2(\mathbb{T}^1)} + \frac{1}{4\varepsilon}\|F\|^2_{L^2(\mathbb{T}^1)}. \tag{3.16}$$

Gathering (3.14), (3.15) and (3.16), one has :

$$\frac{1}{2}\frac{d}{dt}\|\partial_m v\|^2_{L^2(\mathbb{T}^1)} + (\underline{\mathrm{V}} - 2\varepsilon)\int_{\mathbb{T}^1}|\partial_{mm}v|^2 \leqslant \frac{1}{4\varepsilon}\|F\|^2_{L^2(\mathbb{T}^1)} + C_\varepsilon\|\partial_m V\|^4_{L^2(\mathbb{T}^1)}\|\partial_m v\|^2_{L^2(\mathbb{T}^1)}. \tag{3.17}$$

Next, summing (3.13) and (3.17), one obtains :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{H^1(\mathbb{T}^1)} + (\underline{\mathrm{V}} - 2\varepsilon)\|\partial_m v(t)\|^2_{H^1(\mathbb{T}^1)} \leqslant \left(\frac{1}{4\varepsilon}+\frac{1}{2}\right)\|F(t)\|^2_{L^2(\mathbb{T}^1)} \\ + \left(C_\varepsilon\|\partial_m V\|^4_{L^2(\mathbb{T}^1)}+\frac{1}{2}\right)\|v(t)\|^2_{H^1(\mathbb{T}^1)}.\end{aligned}$$

Then, choosing $\varepsilon = \dfrac{\underline{\mathrm{V}}}{4}$, there is a constant C_1 depending only on $\underline{\mathrm{V}}$ such that

$$\frac{d}{dt}\|v(t)\|^2_{H^1(\mathbb{T}^1)} + \underline{\mathrm{V}}\|\partial_m v(t)\|^2_{H^1(\mathbb{T}^1)} \leqslant C_1\|F(t)\|^2_{L^2(\mathbb{T}^1)} + \left(C_1\|\partial_m V\|^4_{L^2(\mathbb{T}^1)}+1\right)\|v(t)\|^2_{H^1(\mathbb{T}^1)}. \tag{3.18}$$

Applying the Grönwall's lemma to (3.18), one obtains :

$$\begin{aligned}\|v\|^2_{\mathscr{C}([0,T],H^1(\mathbb{T}^1))} + \underline{\mathrm{V}}\|\partial_m v\|^2_{L^2((0,T),H^1(\mathbb{T}^1))} \leqslant \left(C_1\|F\|^2_{L^2((0,T)\times\mathbb{T}^1)} + \|v_0\|^2_{H^1(\mathbb{T}^1)}\right) \\ \times \exp\left(T + C_1\|\partial_m V\|^4_{L^4((0,T),L^2(\mathbb{T}^1))}\right).\end{aligned} \tag{3.19}$$

It remains the estimation of $\partial_t v$. Multiplying (3.6) by $\partial_t v$ and integrating in space, one obtains :

$$\begin{aligned}\int_{\mathbb{T}^1}|\partial_t v|^2 + \int_{\mathbb{T}^1}\partial_{tm}vV\partial_m v = \int_{\mathbb{T}^1}\partial_t vF \implies \|\partial_t v\|^2_{L^2(\mathbb{T}^1)} + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^1}V|\partial_x v|^2 \\ = \int_{\mathbb{T}^1}\partial_t vF + \frac{1}{2}\int_{\mathbb{T}^1}|\partial_x v|^2\partial_t V\end{aligned}$$

then

$$\frac{1}{2}\|\partial_t v\|^2_{L^2(\mathbb{T}^1)} + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^1}V|\partial_m v|^2 \leqslant \frac{1}{2}\|F\|^2_{L^2(\mathbb{T}^1)} + \frac{1}{2}\|\partial_m v\|^2_{L^2(\mathbb{T}^1)}\|\partial_t V\|_{L^\infty(\mathbb{T}^1)}. \tag{3.20}$$

Applying again Grönwall's lemma to (3.20), one obtains :

$$\begin{aligned}\|\partial_t v\|^2_{L^2((0,T)\times(\mathbb{T}^1))} + \underline{\mathrm{V}}\|\partial_m v\|^2_{\mathscr{C}([0,T],L^2(\mathbb{T}^1)} \leqslant \left(\|F\|^2_{L^2((0,T)\times\mathbb{T}^1)} + \|V(0)\|_{L^\infty(\mathbb{T}^1)}\|\partial_m v_0\|^2_{L^2(\mathbb{T}^1)}\right) \\ \times \exp\left(\|\partial_t V\|_{L^1((0,T),L^\infty(\mathbb{T}^1))}\right).\end{aligned} \tag{3.21}$$

Finally, gathering (3.19) and (3.21), one has the result. □

Remark 3.1.1. *It is easy to prove that Π_θ given by (3.10) is well defined, linear and continuous. Let us prove that $\mathscr{E}$ defined in (3.11) satisfies $\mathscr{E} = [0,1]$.*

Lemma 3.1.2. $\mathscr{E} = [0,1]$.

Proof. Let us consider the linear non-homogeneous heat equation

$$\begin{cases} \partial_t v - \underline{\mathrm{V}}\partial_{mm} v = F, \\ v(0) = v_0. \end{cases} \tag{3.22}$$

For $F \in L^2((0,T)\times\mathbb{T}^1)$ and $v_0 \in H^2(\mathbb{T}^1)$ there is an unique solution in X of (3.22) and the map $(F, v_0) \in Y \mapsto v \in X$ is continuous. Therefore $\mathscr{E}$ is not empty.

Suppose that $\theta_0 \in \mathscr{E}$, and let us observe that for any $\theta \in [0,1]$:

$$\Pi_\theta = \Pi_{\theta_0} + (\theta-\theta_0)(\Pi_1 - \Pi_0) = \Pi_{\theta_0}\left(I + (\theta-\theta_0)\Pi_{\theta_0}^{-1}(\Pi_1-\Pi_0)\right). \tag{3.23}$$

Thus, for $|\theta-\theta_0| < \|\Pi_{\theta_0}^{-1}(\Pi_1-\Pi_0)\|_{\mathscr{L}(X)}$, one obtains $\theta \in \mathscr{E}$. Consequently, $\mathscr{E}$ is an open set of $[0,1]$.

Let $(\theta_n)_n$ be a sequence of $\mathscr{E}$ that converges to $\theta \in [0,1]$. As (3.23), one has :

$$\Pi_\theta = \Pi_{\theta_n}\left[I + (\theta-\theta_n)\Pi_{\theta_n}^{-1}(\Pi_1-\Pi_0)\right]. \tag{3.24}$$

From the a priori estimates Lemma 3.1.1 and more precisely (3.8), one deduces that $\|\Pi_{\theta_n}^{-1}\|_{\mathscr{L}(Y,X)}$ depends only on $\|V_{\theta_n}(0)\|_{L^\infty(\mathbb{T}^1)}$, $\|\partial_t V_{\theta_n}\|_{L^1(0,T),L^\infty(\mathbb{T}^1))}$ and $\|\partial_m V_{\theta_n}\|_{L^4((0,T),L^2(\mathbb{T}^1))}$ which are bounded uniformly in n, thus

$$K := \sup_n \|\Pi_{\theta_n}^{-1}\|_{\mathscr{L}(Y,X)} < \infty.$$

Let us choose n_0 such that

$$|\theta-\theta_{n_0}|K\|\Pi_0-\Pi_1\|_{\mathscr{L}(Y,X)} < 1,$$

and replace n by n_0 in (3.24), one obtains that $\theta \in \mathscr{E}$, thus $\mathscr{E}$ is a closed set of $[0,1]$.

In short, $\mathscr{E}$ is, at the same time closed, open and non-empty set of $[0,1]$, so $\mathscr{E} = [0,1]$. □

Obviously, Theorem 3.1.4 is a consequence of the fact that $1 \in \mathscr{E}$. □

Proof of Theorem 3.1.5. Assumptions on F and v_0, ensure, thanks to Theorem 3.1.4, the existence of a unique solution of (3.6) satisfying (3.7). By differentiating (3.6) with respect to m, we see that $\partial_m v$ satisfies the following Cauchy problem

$$\begin{cases} \partial_t\partial_m v - \partial_m(V\partial_{mm}v) = \partial_m F - \partial_{mm}V\partial_m v - \partial_m V\partial_{mm} v := \widetilde{F}, \\ \partial_m v(0) = \partial_m v_0. \end{cases} \tag{3.25}$$

Assumptions on F and regularity on v given by Theorem 3.1.4, that is to say $v \in \mathscr{C}([0,T], H^1(\mathbb{T}^1)) \cap L^2((0,T), H^2(\mathbb{T}^1))$, ensure that $\widetilde{F} \in L^2((0,T)\times\mathbb{T}^1)$. Applying again Theorem 3.1.4, we conclude, by uniqueness Theorem 3.1.6, that $\partial_m v \in \mathscr{C}([0,T], H^1(\mathbb{T}^1)) \cap L^2((0,T), H^2(\mathbb{T}^1))$ and satisfies

$$\begin{aligned} \|\partial_m v\|^2_{\mathscr{C}([0,T],H^1(\mathbb{T}^1))} + \|\partial_t\partial_m v\|^2_{L^2((0,T)\times\mathbb{T}^1)} + \underline{\mathrm{V}}\|\partial_{mm}v\|^2_{L^2((0,T),H^1(\mathbb{T}^1))} \\ \leqslant C_1\left(\|\partial_m v_0\|^2_{H^2(\mathbb{T}^1)} + \|\widetilde{F}\|^2_{L^2((0,T)\times\mathbb{T}^1)}\right) \end{aligned}$$

where C_1 is given by (3.8). By Hölder inequality, one has :

$$\begin{aligned} \|\widetilde{F}\|_{L^2((0,T)\times\mathbb{T}^1)} \leqslant \|\partial_m F\|_{L^2((0,T)\times\mathbb{T}^1)} + \|\partial_{mm}V\|_{L^\infty((0,T),L^2(\mathbb{T}^1))}\|\partial_m v\|_{L^2((0,T),L^\infty(\mathbb{T}^1))} \\ + \|\partial_m V\|_{L^\infty((0,T)\times\mathbb{T}^1)}\|\partial_{mm}v\|_{L^2((0,T)\times\mathbb{T}^1)}. \end{aligned} \tag{3.26}$$

By Sobolev embedding $H^1(\mathbb{T}^1) \hookrightarrow L^\infty(\mathbb{T}^1)$, there is a constant $C > 0$ such that :

$$\|\partial_m v\|_{L^2((0,T),L^\infty(\mathbb{T}^1))} \leqslant C\|\partial_m v\|_{L^2((0,T),H^1(\mathbb{T}^1))}.$$

Knowing that v satisfies (3.7), one obtains that the two last terms of the right hand side of(3.26) is less than

$$C_1\underline{\mathrm{V}}^{-1}\max\left(\|\partial_m V\|_{L^\infty((0,T)\times\mathbb{T}^1)}; \|\partial_{mm}V\|_{L^\infty((0,T),L^2(\mathbb{T}^1))}\right)\left(\|v_0\|_{H^2(\mathbb{T}^1)} + \|F\|_{L^2((0,T)\times\mathbb{T}^1)}\right).$$

This helps us to obtain the estimation (3.9), with a constant C depending on T and all condition on V. □

3.1.2 Setting tools for the fixed point theorem

In this section we set tools in order to use the fixed point theorem. The transport equation in mass-Lagrangian coordinate, is easily solved by a simple time integration. Indeed function τ given by

$$\tau(t) = \tau_0 + \int_0^t \partial_m v(s)ds \tag{3.27}$$

is the unique solution of the Cauchy problem

$$\begin{cases} \partial_t\tau - \partial_m v = 0, \\ \tau(0) = \tau_0. \end{cases} \tag{3.28}$$

If $v \in L^1((0,T), H^3(\mathbb{T}^1))$ and $\tau_0 \in H^2(\mathbb{T}^1)$ then $\tau \in \mathscr{C}([0,T], H^2(\mathbb{T}^1))$.

Remark 3.1.2. *From the expression of τ, one deduces easily the following bounds :*

$$(\rho_{max} - \alpha_0)^{-1} - \|\partial_m v\|_{L^1((0,T),L^\infty(\mathbb{T}^1))} \leqslant \tau(t,m) \leqslant \alpha_0^{-1} + \|\partial_m v\|_{L^1((0,T),L^\infty(\mathbb{T}^1))},$$

$$\|\tau\|_{\mathscr{C}([0,T],H^2(\mathbb{T}^1))} \leqslant \|\tau_0\|_{H^2(\mathbb{T}^1)} + \|v\|_{L^1((0,T),H^3(\mathbb{T}^1))}.$$

Using the Theorem 3.1.5, one obtains the following :

Proposition 3.1.1. *Assume that $\tau \in \mathscr{C}([0,T], H^2(\mathbb{T}^1))$, $\partial_t\tau \in \mathscr{C}([0,T], H^1(\mathbb{T}^1))$, $v_0 \in H^2(\mathbb{T}^1)$ and that there exists a constant $\beta > 0$ such that*

$$0 < (\rho_{max} - \beta)^{-1} \leqslant \tau(t,m) \leqslant \beta^{-1}. \tag{3.29}$$

Then the linear non homogeneous problem

$$\begin{cases} \partial_t v - \partial_m(\widetilde{\mu}(\tau)\partial_m v) = -\partial_m\widetilde{P}(\tau), \\ v(0) = v_0 \end{cases} \tag{3.30}$$

admits a unique solution $v \in \mathscr{C}([0,T], H^2(\mathbb{T}^1)) \cap L^2((0,T), H^3(\mathbb{T}^1))$ which satisfies (3.7).

The proof of this proposition boils down to proving that $\widetilde{\mu}(\tau)$ and $\partial_m \widetilde{P}(\tau)$ satisfy hypothesis on V and F respectively in the Theorem 3.1.5. We skip it for the sake of brevity.

Next we consider the vector space $E_T = \mathscr{C}([0,T], H^2(\mathbb{T}^1)) \cap L^2((0,T), H^3(\mathbb{T}^1))$ endowed with the norm $\|\cdot\|_T$ defined by :

$$\|v\|_T^2 := \|v\|^2_{\mathscr{C}([0,T],H^2(\mathbb{T}^1))} + \|\partial_m v\|^2_{L^2((0,T),H^2(\mathbb{T}^1))}.$$

Obviously, $(E_T, \|\cdot\|_T)$ is a Banach space, and consequently, any closed ball of E_T is complete. For all $R > 0$ we denote by $E_T(R)$ the closed ball of radius R of E_T. Given $v \in E_T(R)$, we consider τ_v (3.27) the unique solution of (3.28). Bounding $\|\partial_m v\|_{L^1((0,T),L^\infty(\mathbb{T}^1))}$ in the first inequality of Remark 3.1.2 with the Cauchy Schwartz inequality in time, one obtains that there exist $T_1 > 0$ and a constant $\eta = \eta(T_1, R) > 0$ such that, for all $0 \leqslant t \leqslant T_1$,

$$0 < (\rho_{max} - \eta)^{-1} \leqslant \tau(t,m) \leqslant \eta^{-1}.$$

Then, adding Proposition 3.1.1 , one obtains that for any $v \in E_{T_1}(R)$ there is a unique solution w of the Cauchy problem (3.30), with τ given by (3.27). In the following Proposition, we show that if T is small, then $w \in E_T(R)$.

Proposition 3.1.2. *There are $R^* > 0$ depending on the initial data and $T_2 \leqslant T_1$ inversely proportional to R^* such that $\|w\|_{E_{T_2}} \leqslant R^*$.*

Remark 3.1.3. *Adding Proposition 3.1.2 to our previous analysis, one concludes the fact that the map*

$$\begin{aligned} \Phi_{T_2} \colon E_{T_2}(R^*) &\to E_{T_2}(R^*) \\ v &\mapsto w \end{aligned}$$

where w the unique solution of (3.30) with τ given by (3.27), is well defined.

It remains to show that Φ_T is a contraction for some small enough $T \leqslant T_2$. This result is stated in the following.

Proposition 3.1.3. *There exists T^* small and $0 < \kappa < 1$ such that for any $v, w \in E_{T^*}(R^*)$ we have*

$$\|\Phi_{T^*}(v) - \Phi_{T^*}(w)\|_{E_{T^*}} \leqslant \kappa \|v - w\|_{E_{T^*}}.$$

Using the fixed point theorem, we conclude that there exists a unique $v_* \in E_{T^*}(R)$ such that $\Phi_{T^*}(v_*) = v_*$. Such v_* and τ_* given by

$$\tau_*(t) = \tau_0 + \int_0^t \partial_m v_*(s) ds,$$

satisfy the Navier Stokes system (3.3) in classical sense.

Proof of Proposition 3.1.2. Let us set $V = \widetilde{\mu}(\tau)$ and $F = -\partial_m \widetilde{P}(\tau)$ and rewrite the estimation for w.

$$\|w\|^2_{\mathscr{C}([0,T],H^2(\mathbb{T}^1))} + \|\partial_t\|^2_{L^2((0,T),H^1(\mathbb{T}^1))} + \underline{V}\|\partial_m w\|^2_{L^2((0,T)\times H^2(\mathbb{T}^1))} \\ \leqslant C\left(\|v_0\|^2_{H^2(\mathbb{T}^1)} + \|F\|^2_{L^2((0,T),H^1(\mathbb{T}^1))}\right) \ : \quad (3.31)$$

with $\underline{\mathrm{V}}$ depending on η. As shown before, the constant C is given by

$$C = C_1 \left(1 + \underline{\mathrm{V}}^{-1} \max\left(\|\partial_m V\|_{L^\infty((0,T)\times(\mathbb{T}^1))}; \|\partial_{mm} V\|_{L^\infty((0,T),L^2(\mathbb{T}^1))}\right)\right) \tag{3.32}$$

where C_1 is given by (3.8). Remembering that $v \in E_T(R)$, and assuming that $T < T_1$ one has the following estimates:

$$\|V(0)\|_{L^\infty(\mathbb{T}^1)} = c_1(\alpha); \quad \|\partial_t V\|_{L^1((0,T),L^\infty(\mathbb{T}^1))} \leqslant c_2(\eta) RT;$$

$$\|\partial_m V\|_{L^4((0,T),L^2(\mathbb{T}^1))} \leqslant c_3(\eta) T^{1/4} \left(\|\partial_m \tau_0\|_{L^2(\mathbb{T}^1)} + RT\right);$$
$$\|\partial_m V\|_{L^\infty((0,T)\times(\mathbb{T}^1))} \leqslant c_4(\eta) \left(\|\partial_m \tau_0\|_{L^2((\mathbb{T}^1))} + RT\right);$$
$$\|\partial_{mm} V\|_{L^\infty((0,T),L^2(\mathbb{T}^1))} \leqslant c_5(\eta) \left(\|\partial_{mm} \tau_0\|_{L^2(\mathbb{T}^1)} + RT^{1/2}\right);$$
$$\|F\|_{L^2(0,T,H^1(\mathbb{T}^1))} \leqslant c_6(\eta) T^{\frac{1}{2}} (\|\tau_0\|_{H^2(\mathbb{T}^1)} + R\sqrt{T})(2 + \|\tau_0\|_{H^2(\mathbb{T}^1)} + R\sqrt{T}).$$

Let us remark that η is supposed to be fixed, thus constants c_i do not depend on T neither R. Thus, choosing a large R depending on norm of the initial data and a small T less than a constant times the inverse of some power of R, one can make the right hand side of (3.31) less than R, which completes the proof. □

Proof of Proposition 3.1.3. Let $v, w \in E_{T^*}(R^*)$, $\overline{u} = \Phi_{T^*}(u)$, $\overline{v} = \Phi_{T^*}(v)$ and

$$\tau_u(t) = \tau_0 + \int_0^t \partial_m u(s) ds, \quad \tau_v(t) = \tau_0 + \int_0^t \partial_m v(s) ds.$$

As $\overline{u}$, $\overline{v}$ satisfy (3.30) with τ_u and τ_v respectively, one has

$$\begin{cases} \partial_t (\overline{u} - \overline{v}) - \partial_m (\widetilde{\mu}(\tau_u) \partial_m (\overline{u} - \overline{v})) = -F, \\ (\overline{u} - \overline{v})(0) = 0 \end{cases} \tag{3.33}$$

with F given by

$$F = \underbrace{\widetilde{P}'(\tau_u)\partial_m \tau_u - \widetilde{P}'(\tau_v)\partial_m \tau_v}_{F_1} - \underbrace{\partial_m \left\{\left(\widetilde{\mu}(\tau_u) - \widetilde{\mu}(\tau_v)\right) \partial_m \overline{v}\right\}}_{F_2}.$$

Noting that F_1 can be written in the form

$$F_1 = \widetilde{P}'(\tau_u) \left(\partial_m \tau_u - \partial_m \tau_v\right) + \left(\widetilde{P}'(\tau_u) - \widetilde{P}'(\tau_v)\right) \partial_m \tau_v,$$

one easily obtains

$$\begin{aligned} \|F_1\|_{L^2((0,T)\times\mathbb{T}^1)} \leqslant c_7(\eta) T^{3/2} \big[\|\partial_{mm}(u - v)\|_{\mathscr{C}([0,T],L^2(\mathbb{T}^1))} + \left(\|\tau_0 - \overline{\tau}\|_{H^2(\mathbb{T}^1)} + RT\right) \\ \|\partial_m (u - v)\|_{\mathscr{C}([0,T],L^2(\mathbb{T}^1))}\big] \leqslant c_7(\eta) T^{3/2} \left(1 + \|\tau_0\|_{H^2(\mathbb{T}^1)} + RT\right) \|u - v\|_{E_{T^*}}. \end{aligned} \tag{3.34}$$

Putting $\partial_m F_1$ in the form

$$\begin{aligned} \partial_m F_1 = \widetilde{P}''(\tau_u)(\partial_m \tau_u + \partial_m \tau_v)(\partial_m \tau_u - \partial_m \tau_v) + \left(\widetilde{P}''(\tau_u) - \widetilde{P}''(\tau_v)\right) \\ + \widetilde{P}'(\tau_u)\partial_{mm}(\tau_u - \tau_v) + \left(\widetilde{P}'(\tau_u) - \widetilde{P}'(\tau_v)\right) \partial_{mm} \tau_v, \end{aligned}$$

one shows that

$$\|\partial_m F_1\|_{L^2((0,T)\times\mathbb{T}^1)} \leqslant c_8(\eta)T^{3/2}\left[1+\|\tau_0\|_{H^2(\mathbb{T}^1)}+R\sqrt{T}\right]\|u-v\|_{E_{T^*}}. \tag{3.35}$$

We turn now to the estimation of F_2. We rewrite F_2 on the form

$$F_2 = \left[\widetilde{\mu}'(\tau_u)\left(\partial_m(\tau_u-\tau_v)\right)+\left(\widetilde{\mu}'(\tau_u)-\widetilde{\mu}'(\tau_v)\right)\partial_m\tau_v\right]\partial_m\overline{v}+\left(\widetilde{\mu}(\tau_u)-\widetilde{\mu}(\tau_v)\right)\partial_{mm}\overline{v}.$$

And one finds easily

$$\|F_2\|_{L^2((0,T)\times\mathbb{T}^1)} \leqslant c_9(\eta)RT^{3/2}\left(1+\|\tau_0\|_{H^2(\mathbb{T}^1)}+TR\right)\|u-v\|_{E_{T^*}}. \tag{3.36}$$

Despite the fact that $\partial_m F_2$ seems a little bit difficult to estimate, as before, we put it on the form

$$\begin{aligned}\partial_m F_2 = &\left[\widetilde{\mu}''(\tau_u)\left(\partial_m\tau_u+\partial_m\tau_v\right)\left(\partial_m\tau_u-\partial_m\tau_v\right)+\left(\widetilde{\mu}''(\tau_u)-\widetilde{\mu}''(\tau_v)\right)|\partial_m\tau_v|^2\right.\\ &\left.+\widetilde{\mu}'(\tau_u)\left(\partial_{mm}\tau_u-\partial_{mm}\tau_v\right)+\left(\widetilde{\mu}'(\tau_u)-\widetilde{\mu}'(\tau_v)\right)\partial_{mm}\tau_v\right]\partial_m\overline{v}\\ &+\left[\widetilde{\mu}'(\tau_u)\left(\partial_m\tau_u-\partial_m\tau_v\right)+\left(\widetilde{\mu}'(\tau_u)-\widetilde{\mu}'(\tau_v)\right)\partial_m\tau_v\right]\partial_{mm}\overline{v}+\partial_{mmm}\overline{v}\left(\widetilde{\mu}(\tau_u)-\widetilde{\mu}(\tau_v)\right).\end{aligned}$$

And one obtains,

$$\begin{aligned}\|\partial_m F_2\|_{L^2((0,T)\times\mathbb{T}^1)} \leqslant c_{10}(\eta)TR&\left[1+R+\sqrt{T}\left(\|\tau_0\|_{H^2(\mathbb{T}^1)}+R\sqrt{T}\right)\right.\\ &\left.\left(1+R+\|\tau_0\|_{H^2(\mathbb{T}^1)}+R\sqrt{T}\right)\right]\|u-v\|_{E_{T^*}}.\end{aligned} \tag{3.37}$$

As $\overline{u}-\overline{v}$ is solution of (3.33), then thanks to Theorem 3.1.5, it satisfies (3.9), with $v_0=0$, a constant C given by (3.32). Adding estimations (3.34), (3.35), (3.36) and (3.37) and choosing a small T^* inversely proportional to a power of R, one obtains that Φ_{T^*} is a contracting map. This ends the proof. □

3.1.3 Back to *Eulerian* coordinate

Above, we proved that given $(\rho_0,u_0)\in H^2(\mathbb{T}^1)\times H^2(\mathbb{T}^1)$, there exists a classical solution (τ_*,v_*) of the Cauchy problem associated to (3.3) with initial data (τ_0,v_0) defined by :

$$\tau_0(m)=1/\rho_0(Y(m)) \text{ and } v_0=u_0(Y(m)).$$

As we showed in section 2.2, it is obvious that (ρ_*,u_*) defined by :

$$u_*(t,x)=v_*(t,m(t,x)) \quad \text{and} \quad \rho_*(t,x)=\frac{1}{\tau_*(t,m(t,x))}$$

is solution of the Navier Stokes system (3.1)in *Eulerian* coordinate. In the following we prove that (ρ_*,u_*) is as regular as (τ_*,v_*).

Let us note that the jacobian of the change of variable $J\colon (t,x)\mapsto(t,m(t,x))$ is $1/\tau_*$. Then, as τ_* is far from zero, $J\in\mathscr{C}([0,T],H^2(\mathbb{T}^1))$ and consequently $(\rho_*,v_*)\in\left(\mathscr{C}([0,T],H^2(\mathbb{T}^1))\right)^2$. It remains to prove that $\partial_{xxx}u_*\in L^2((0,T)\times\mathbb{T}^1)$. First, one uses the fact that $\dfrac{\partial x}{\partial m}=\tau_*$ to obtain

$$\partial_{xxx}u_*=\tau_*|\partial_m\tau_*|^2\partial_m v_*+\tau_*^2\partial_{mm}\tau_*\partial_m v_*+3\tau_*^2\partial_m\tau_*\partial_{mm}v_*+\tau_*^3\partial_{mmm}v_*.$$

Then using the regularity on τ_*, v_* and the fact that τ is far from zero, one proves that $\partial_{xxx}u_*\in L^2((0,T)\times\mathbb{T}^1)$. And the result follows.

3.2 Uniqueness of the local solution

In this section we prove that the solution that we constructed with a fixed point theorem is unique. The statement is the following.

Theorem 3.2.1. *Given $(\rho_0, u_0) \in H^2(\mathbb{T}^1) \times H^2(\mathbb{T}^1)$. Then for any $T > 0$ there is at most one solution $(\rho, u) \in \mathscr{C}([0,T], H^2(\mathbb{T}^1)) \times (\mathscr{C}([0,T], H^2(\mathbb{T}^1)) \cap L^2((0,T), H^3(\mathbb{T}^1)))$ of the Cauchy problem*

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + P(\rho) - \mu(\rho)\partial_x u) = 0, \\ (\rho(0), u(0)) = (\rho_0, u_0) \end{cases} \tag{3.38}$$

such that there exists $\boldsymbol{c} = \boldsymbol{c}(T) > 0$: $\boldsymbol{c} \leqslant \rho(t,x) \leqslant \rho_{max} - \boldsymbol{c}$.

Remark 3.2.1. *Let (ρ, u) and (ρ', u') be two solutions of the Navier Stokes system (3.1), and let us define $\delta\rho = \rho - \rho'$ and $\delta u = u - u'$. We can easily convince ourselves that the proof of Theorem 3.2.1 comes from the following theorem :*

Theorem 3.2.2. *There exist two positives constants $C = C(T, \rho, \rho', u, u', \bar{\rho})$ and $\nu = \nu(\boldsymbol{c})$ such that*

$$\sup_{0 \leqslant t \leqslant T} \int_{\mathbb{T}^1} \left[\rho(t,x)|\delta u(t,x)|^2 + |\delta\rho(t,x)|^2\right] dx + \nu \int_0^T \int_{\mathbb{T}^1} |\partial_x \delta u(t,x)|^2 dt dx$$
$$\leqslant C \int_{\mathbb{T}^1} \left[\rho(t,x)|\delta u(0,x)|^2 + |\delta\rho(0,x)|^2\right] dx. \tag{3.39}$$

Proof of Theorem 3.2.2. Let us note that $\delta\rho$ and δu satisfy :

$$\begin{cases} \partial_t \delta\rho + \partial_x (u\delta\rho) + \partial_x (\rho'\delta u) = 0, \\ \rho\partial_t \delta u + \delta\rho\partial_t u' + \rho\partial_x \left(\delta u \dfrac{u+u'}{2}\right) + \delta\rho\partial_x \dfrac{|u'|^2}{2} + P'(\rho)\partial_x\delta\rho \\ + (P'(\rho) - P'(\rho'))\,\partial_x\rho' - \partial_x (\mu(\rho)\partial_x\delta u) - \partial_x ((\mu(\rho) - \mu(\rho'))\partial_x u') = 0. \end{cases} \tag{3.40}$$

Now, multiply the first equation of (3.40) by $\delta\rho$, one obtains, for $\varepsilon > 0$,

$$\frac{1}{2}\frac{d}{dt}\|\delta\rho(t)\|^2_{L^2(\mathbb{T}^1)} \leqslant \frac{1}{2}\|\delta\rho(t)\|^2_{L^2(\mathbb{T}^1)}\|\partial_x u(t)\|_{L^\infty(\mathbb{T}^1)} + 2\varepsilon\|\partial_x\delta u(t)\|^2_{L^2(\mathbb{T}^1)}$$
$$+ c_\varepsilon\left[\|\delta u(t)\|^2_{L^2(\mathbb{T}^1)}\|\partial_x\rho'\|^2_{L^2(\mathbb{T}^1)} + \|\delta\rho\|^2_{L^2(\mathbb{T}^1)}\|\partial_x\rho'\|_{L^2(\mathbb{T}^1)}\right] + \frac{1}{4\varepsilon}\|\delta\rho(t)\|^2_{L^2(\mathbb{T}^1)}\|\rho'\|^2_{L^\infty(\mathbb{T}^1)}. \tag{3.41}$$

Next multiply the second equation of (3.40) by δu, one obtains that there exists a constant C_1 depending only on $\mathbf{c}$ and ε such that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^1}\rho|\delta u(t)|^2 + \int_{\mathbb{T}^1}(\mu(\rho) - 2\varepsilon)\|\partial_x\delta u(t)\|^2 \leqslant C_1\left(\|\delta u\|^2_{L^2(\mathbb{T}^1)} + \|\delta\rho\|^2_{L^2(\mathbb{T}^1)}\right)$$
$$\Big(1 + \|\partial_t\rho\|_{L^\infty(\mathbb{T}^1)} + \|\partial_t u'\|_{L^\infty(\mathbb{T}^1)} + \|\rho\partial_x(u+u') - (u+u')\partial_x\rho\|_{L^\infty(\mathbb{T}^1)} + \|\partial_x u'^2\|_{L^\infty(\mathbb{T}^1)}$$
$$+\|\partial_x\rho\|_{L^\infty(\mathbb{T}^1)} + \|\partial_x\rho'\|_{L^\infty(\mathbb{T}^1)} + \|\partial_x u'\|_{L^\infty(\mathbb{T}^1)}\Big). \tag{3.42}$$

Summing (3.41) and (3.42), one easily apply Grönwall lemma and obtains (3.39). Terms in the obtained inequality after applying Grönwall inequality coming from blue terms in (3.41) and (3.42) are easily bounded by a constant depending on norms of T, ρ, ρ', u and u' maybe except $\|\partial_t\rho\|_{L^1((0,T),L^\infty(\mathbb{T}^1))}$ and $\|\partial_t u'\|_{L^1((0,T),L^\infty(\mathbb{T}^1))}$. But by (3.38), one has $\partial_t\rho \in \mathscr{C}([0,T], H^1(\mathbb{T}^1))$ and because ρ' is far from vacuum, $\partial_t u' \in L^2((0,T), H^1(\mathbb{T}^1))$. This ends the proof. $\square$

3.3 Blow up criterion

In this section, we want to find a criterion for which the solution of the Cauchy problem associated to the Navier Stokes system with initial data $(\rho_0, u_0) \in H^2(\mathbb{T}^1) \times H^2(\mathbb{T}^1)$ is globally defined. Let us note that the time of existence of the local solution is proportional to the inverse of a power of the norm of initial data, which explains the fact that the smaller the initial data, the greater the time of existence. Throughout this section the viscosity take the form $\mu(\rho) = \rho^\theta$ with $\theta \geqslant 0$ and the pressure P satisfies (3.2). In the following lines, we state an extension theorem for our system which is similar to the classical one for ODEs.

Theorem 3.3.1. *Let $(\tau, v) \in \mathscr{C}([0,T], H^2(\mathbb{T}^1)) \times (\mathscr{C}([0,T], H^2(\mathbb{T}^1)) \cap L^2((0,T), H^3(\mathbb{T}^1)))$ be the local solution of (3.3) with initial data $(\tau_0, v_0) \in H^2(\mathbb{T}^1) \times H^2(\mathbb{T}^1)$. Assume that there are two positives constants $C(T)$ and $\boldsymbol{c}(T)$ such that*

$$\forall\, 0 \leqslant t < T \quad \|\tau(t)\|_{H^2(\mathbb{T}^1)}, \|u(t)\|_{H^2(\mathbb{T}^1)} \leqslant C \quad \text{and} \quad (\rho_{max} - \boldsymbol{c})^{-1} \leqslant \tau(t) \leqslant \boldsymbol{c}^{-1}.$$

Then the solution can be extended beyond T.

The proof of this theorem is based on the fact that one can construct one solution of the same system starting very closely to T.

From this theorem, one has the following blow-up criterion :

Remark 3.3.1. *Let T^* be the maximal existence time of the local solution (τ, v) of the Navier Stokes system (3.3) with initial data $(\tau_0, v_0) \in H^2(\mathbb{T}^1) \times H^2(\mathbb{T}^1)$. Assume that there are two non-negative constants $C > 0$, $\boldsymbol{c} > 0$ such that*

$$\forall\, 0 \leqslant t < T^*, \quad \|\tau(t)\|_{H^2(\mathbb{T}^1)}, \|u(t)\|_{H^2(\mathbb{T}^1)} \leqslant C \quad \text{and} \quad (\rho_{max} - \boldsymbol{c})^{-1} \leqslant \tau(t) \leqslant \boldsymbol{c}^{-1}. \tag{3.43}$$

Then $T^* = +\infty$.

The new challenge is to show that the bound on the density controls the higher Sobolev norms of the solution. Consequently, the previous explosion criterion could be simplified by taking into account only the fact that the density is away from zero and ρ_{max}. The result is stated in the following theorem :

Theorem 3.3.2. *Let T^* be the maximal existence time of the local solution (τ, v) of the Navier Stokes system (3.3) with initial data $(\tau_0, v_0) \in H^2(\mathbb{T}^1) \times H^2(\mathbb{T}^1)$. Assume that*

$$0 < \inf_{[0,T^*)\times\mathbb{T}^1} \tau \leqslant \sup_{[0,T^*)\times\mathbb{T}^1} \tau < \rho_{max}. \tag{3.44}$$

Then $T^* = +\infty$.

To obtain the above theorem we will prove Remark 3.3.1 : we only need to estimate the higher Sobolev norms of τ and v. We first obtain $\partial_m v \in L^2((0,T), L^\infty(\mathbb{T}^1))$, which allows us to close estimates for τ and v. The proof of the blow-up criterion in *Eulerian coordinate* Theorem 3.1.3 follows. Indeed, ρ bounded in *Eulerian* coordinate is equivalent to ρ bounded in mass-Lagrangian coordinate with the same bounds.

Proof of Theorem 3.3.1. Let be $t_0 = T - \varepsilon C^{-1} > 0$ with ε small and consider the Cauchy problem associated to (3.3) with initial data $(\tau(t_0), v(t_0))$. Then, according to the previous section, there exists a unique solution valid on (t_0, T^*) with $T^* - t_0$ greater than a constant times C^{-1}. Then the smallness of ε allows $T^* - T$ to be non negative otherwise like a constant times C^{-1}. By uniqueness, on $[t_0, T]$ the new solution equals the first one. And by juxtaposing the two solutions, one obtains a new one on $[0, T^*]$. □

Proof of Theorem 3.3.2. Multiplying the first equation of (3.3) by τ the second one by v, summing and integrating on $\mathbb{T}^1$ and using a Grönwall lemma one has the following estimates :

Lemma 3.3.1. *There exists* $C_1 = C_1(T, \mathbf{c}, \|\tau_0\|_{L^2(\mathbb{T}^1)}, \|v_0\|_{L^2(\mathbb{T}^1)})$ *such that*

$$\sup_{0\leqslant t\leqslant T}\{\|\tau(t)\|^2_{L^2(\mathbb{T}^1)} + \|v(t)\|^2_{L^2(\mathbb{T}^1)}\} + \int_0^T\int_{\mathbb{T}^1}|\partial_m v(t,m)|^2 dtdm \leqslant C_1. \tag{3.45}$$

In the following, we obtain $\partial_m v \in L^2((0,T), L^\infty(\mathbb{T}^1))$.

Lemma 3.3.2. *There exists* $C_2 = C_2(T, \mathbf{c}, \|\widetilde{\rho}_0\|_{L^2(\mathbb{T}^1)}, \|v_0\|_{H^1(\mathbb{T}^1)})$ *such that*

$$\int_0^T\int_{\mathbb{T}^1}|v(t,m)|^2 dtdm + \sup_{0\leqslant t\leqslant T}\int_{\mathbb{T}^1}|\partial_m v(t,m)|^2 dm \leqslant C_2. \tag{3.46}$$

Moreover, there exists $C_3 = C_3(T, \mathbf{c}, \|\widetilde{\rho}_0\|_{L^2(\mathbb{T}^1)}, \|v_0\|_{H^1(\mathbb{T}^1)})$ *such that*

$$\int_0^T \|\partial_m v(t)\|^2_{L^\infty(\mathbb{T}^1)} dt \leqslant C_3.$$

Proof. Let us multiply the momentum equation by $\partial_t v$ and integrate on $\mathbb{T}^1$. One obtains :

$$\int_{\mathbb{T}^1}|\partial_t v|^2 + \frac{1}{2}\int_{\mathbb{T}^1}\widetilde{\rho}^{1+\theta}\partial_t|\partial_m v|^2 = \int_{\mathbb{T}^1}\partial_{tm}vP(\widetilde{\rho}).$$

First,

$$\int_{\mathbb{T}^1}\widetilde{\rho}^{1+\theta}\partial_t|\partial_m v|^2 = \frac{d}{dt}\int_{\mathbb{T}^1}\widetilde{\rho}^{1+\theta}|\partial_m v|^2 + (1+\theta)\int_{\mathbb{T}^1}\widetilde{\rho}^{2+\theta}(\partial_m v)^3$$

and

$$\int_{\mathbb{T}^1}\partial_{tm}vP(\widetilde{\rho}) = \frac{d}{dt}\int_{\mathbb{T}^1}\partial_m vP(\widetilde{\rho}) + \int_{\mathbb{T}^1}P'(\widetilde{\rho})\widetilde{\rho}^2|\partial_m v|^2.$$

Then gathering the three above equations, one has :

$$\int_{\mathbb{T}^1}|\partial_t v|^2 + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^1}\widetilde{\rho}^{1+\theta}|\partial_m v|^2 = \frac{d}{dt}\int_{\mathbb{T}^1}\partial_m vP(\widetilde{\rho}) + \int_{\mathbb{T}^1}P'(\widetilde{\rho})\widetilde{\rho}^2|\partial_m v|^2 - \frac{1+\theta}{2}\int_{\mathbb{T}^1}\widetilde{\rho}^{2+\theta}(\partial_m v)^3.$$

Integrating the above equation on $(0,t)$ with respect to time, one has :

$$\begin{aligned}\int_0^t\int_{\mathbb{T}^1}|\partial_t v|^2 + \frac{1}{2}\int_{\mathbb{T}^1}\widetilde{\rho}^{1+\theta}|\partial_m v(t)|^2 &= \frac{1}{2}\int_{\mathbb{T}^1}\widetilde{\rho}_0^{1+\theta}|\partial_m v_0|^2 + \int_{\mathbb{T}^1}\partial_m v(t)P(\widetilde{\rho}(t)) - \int_{\mathbb{T}^1}\partial_m v_0 P(\widetilde{\rho}_0)\\ &\quad + \int_0^t\int_{\mathbb{T}^1}\widetilde{\rho}^2|\partial_m v|^2 P'(\widetilde{\rho}) - \frac{1+\theta}{2}\int_0^t\int_{\mathbb{T}^1}\widetilde{\rho}^{2+\theta}(\partial_m v)^3. \end{aligned}\tag{3.47}$$

In order to apply Grönwall lemma one has to estimate precisely each term appearing in the right hand side of the above equation. Note that the singularity of the pressure makes the task complicated, especially in terms showing $P(\widetilde{\rho})$ or $P'(\widetilde{\rho})$, for instance when ρ becomes closer and closer to ρ_{max}. This justifies the fact that the assumption of uniform bounds on the density (3.44) is of paramount importance. It implies in particular that $P(\widetilde{\rho})$ and $P'(\widetilde{\rho})$ are $L^\infty((0,T)\times\mathbb{T}^1)$ and by Hölder inequality and Lemma 3.3.1, one has, for some $\varepsilon > 0$:

$$\left|\int_{\mathbb{T}^1} \partial_m v(t) P(\widetilde{\rho}(t))\right| \leqslant \varepsilon \int_{\mathbb{T}^1} \widetilde{\rho}^{1+\theta}(t)|\partial_m v(t)|^2 + \frac{1}{4\varepsilon}\int_{\mathbb{T}^1} \widetilde{\rho}(t)^{-(1+\theta)} P(\widetilde{\rho}(t)); \tag{3.48}$$

$$\left|\int_0^t \int_{\mathbb{T}^1} \widetilde{\rho}^2 |\partial_m v|^2 P'(\widetilde{\rho})\right| \leqslant \int_0^t \|P'(\widetilde{\rho})\widetilde{\rho}^{1-\theta}\|_{L^\infty(\mathbb{T}^1)} \int_{\mathbb{T}^1} \widetilde{\rho}^{1+\theta}|\partial_m v(t)|^2. \tag{3.49}$$

The term $\displaystyle\int_0^T\int_{\mathbb{T}^1} \widetilde{\rho}^{2+\theta}(\partial_m v)^3$ forces a $L^2((0,T),L^\infty(\mathbb{T}^1))$ bound on $\partial_m v$. To prove this under the assumption (3.44), we will use the bound on $\widetilde{\rho}^{1+\theta}\partial_m v - P(\widetilde{\rho})$ which is nothing other than the *effective flux* in mass Lagrangian coordinates.

$$\begin{aligned}
\left|\int_0^t\int_{\mathbb{T}^1} \widetilde{\rho}^{2+\theta}(\partial_m v)^3\right| &\leqslant \left|\int_0^t\int_{\mathbb{T}^1} \widetilde{\rho}|\partial_m v|^2\left(\widetilde{\rho}^{1+\theta}\partial_m v - P(\widetilde{\rho})\right)\right| + \left|\int_0^t\int_{\mathbb{T}^1}\widetilde{\rho}|\partial_m v|^2 P(\widetilde{\rho})\right| \\
&\leqslant \int_0^t \|\widetilde{\rho}^{1+\theta}\partial_m v - P(\widetilde{\rho})\|_{L^\infty(\mathbb{T}^1)}\int_{\mathbb{T}^1}\widetilde{\rho}|\partial_m v|^2 + \int_0^t \|\widetilde{\rho}^{-\theta}P(\widetilde{\rho})\|_{L^\infty(\mathbb{T}^1)}\int_{\mathbb{T}^1}\widetilde{\rho}^{1+\theta}|\partial_m v|^2 \\
&\leqslant \frac{1}{2}\int_0^t \|\widetilde{\rho}^{1+\theta}\partial_m v - P(\widetilde{\rho})\|^2_{L^\infty(\mathbb{T}^1)} + \frac{1}{2}\int_0^t\left[\int_{\mathbb{T}^1}\widetilde{\rho}|\partial_m v|^2\right]^2 \\
&\quad + \int_0^t \|\widetilde{\rho}^{-\theta}P(\widetilde{\rho})\|_{L^\infty(\mathbb{T}^1)}\int_{\mathbb{T}^1}\widetilde{\rho}^{1+\theta}|\partial_m v|^2.
\end{aligned}$$

By Gagliardo-Niremberg inequality, one has :

$$\|\widetilde{\rho}^{1+\theta}\partial_m v - P(\widetilde{\rho})\|^2_{L^\infty} \leqslant 2\|\partial_t v\|_{L^2(\mathbb{T}^1)}\|\widetilde{\rho}^{1+\theta}\partial_m v - P(\widetilde{\rho})\|_{L^2(\mathbb{T}^1)} \leqslant \|\partial_t v\|^2_{L^2(\mathbb{T}^1)} + \|\widetilde{\rho}^{1+\theta}\partial_m v - P(\widetilde{\rho})\|^2_{L^2(\mathbb{T}^1)}$$

then

$$\begin{aligned}
\left|\int_0^t\int_{\mathbb{T}^1}\widetilde{\rho}^{2+\theta}(\partial_m v)^3\right| \leqslant \frac{1}{2}\int_0^t\int_{\mathbb{T}^1}|\partial_t v|^2 + \int_0^t \|P(\widetilde{\rho})\|^2_{L^2(\mathbb{T}^1)} + \int_0^t \Big[\|\widetilde{\rho}^{1+\theta}\|_{L^\infty(\mathbb{T}^1)} + \|\widetilde{\rho}^{-\theta}P(\widetilde{\rho})\|_{L^\infty(\mathbb{T}^1)} \\
+\|\widetilde{\rho}^{1-\theta}\|_{L^\infty(\mathbb{T}^1)}\left[\int_{\mathbb{T}^1}|\partial_m v|^2\right]\Big]\int_{\mathbb{T}^1}\widetilde{\rho}^{1+\theta}|\partial_m v|^2.
\end{aligned} \tag{3.50}$$

Gathering (3.47), (3.48), (3.49) and (3.50), one obtains :

$$\begin{aligned}
\int_0^t\int_{\mathbb{T}^1}|\partial_t v|^2 + \frac{1}{2}\int_{\mathbb{T}^1}\widetilde{\rho}^{1+\theta}|\partial_m v(t)|^2 &\leqslant \frac{1}{2}\int_{\mathbb{T}^1}\widetilde{\rho}(t)^{-(1+\theta)}P(\widetilde{\rho}(t)) + \int_0^t\|P(\widetilde{\rho})\|^2_{L^2(\mathbb{T}^1)} \\
&\quad + (1+\theta)\int_0^t\Big[\|\widetilde{\rho}^{1+\theta}\|_{L^\infty(\mathbb{T}^1)} + \|\widetilde{\rho}^{-\theta}P(\widetilde{\rho})\|_{L^\infty(\mathbb{T}^1)} + \|P'(\widetilde{\rho})\widetilde{\rho}^{1-\theta}\|_{L^\infty(\mathbb{T}^1)} \\
&\quad + \|\widetilde{\rho}^{1-\theta}\|_{L^\infty(\mathbb{T}^1)}\left[\int_{\mathbb{T}^1}|\partial_m v|^2\right]\Big]\int_{\mathbb{T}^1}\widetilde{\rho}^{1+\theta}|\partial_m v|^2 \\
&\quad + \left|\frac{1}{2}\int_{\mathbb{T}^1}\widetilde{\rho}_0^{1+\theta}|\partial_m v_0|^2 - \int_{\mathbb{T}^1}\partial_m v_0 P(\widetilde{\rho}_0)\right|.
\end{aligned}$$

Then, applying the Grönwall lemma, the boundedness of the density and the estimates Lemma 3.3.1, one has the following bound with a constant $C = C(\mathbf{c}, T, E_0, \|\partial_m v_0\|_{L^2(\mathbb{T}^1)})$,

$$\int_0^T \int_{\mathbb{T}^1} |\partial_t v|^2 + \sup_{0\leqslant t\leqslant T} \int_{\mathbb{T}^1} \widetilde{\rho}^{1+\theta} |\partial_m v(t)|^2 \leqslant C.$$

This completes the first part of the lemma. The second one comes from the following :

$$\begin{aligned}\int_0^T \|\partial_m v(t)\|^2_{L^\infty(\mathbb{T}^1)} &\leqslant 2\int_0^T \|\partial_m v - P(\rho)\|^2_{L^\infty(\mathbb{T}^1)} + \int_0^T \|P(\rho)\|^2_{L^\infty(\mathbb{T}^1)}\\ &\leqslant 4\int_0^T \|\partial_m v - P(\rho)\|_{L^2(\mathbb{T}^1)} \|\partial_t v\|_{L^2(\mathbb{T}^1)} + \int_0^T \|P(\rho)\|^2_{L^\infty(\mathbb{T}^1)}\\ &\leqslant 2\int_0^T \|\partial_t v\|^2_{L^2(\mathbb{T}^1)} + 4\left(\int_0^T \|\partial_m v\|^2_{L^2(\mathbb{T}^1)} + \|P(\rho)\|^2_{L^2(\mathbb{T}^1)}\right) + \int_0^T \|P(\rho)\|^2_{L^\infty(\mathbb{T}^1)} \square\end{aligned}$$

Using Remark 3.3.1, one notes that the proof of Theorem 3.3.2 is achieved after proving the following :

Lemma 3.3.3. *Given $(\tau, v) \in \mathscr{C}([0,T], H^2(\mathbb{T}^1)) \times (\mathscr{C}([0,T], H^2(\mathbb{T}^1)) \cap L^2((0,T), H^3(\mathbb{T}^1)))$ solution of the Cauchy problem associated to the Navier Stokes system (3.3) with initial data $(\tau_0, v_0) \in H^2(\mathbb{T}^1) \times H^2(\mathbb{T}^1)$. Then there exist constant $C = C(T, \mathbf{c}, \|\tau_0\|_{H^2(\mathbb{T}^1)}, \|v_0\|_{H^2(\mathbb{T}^1)})$ and $\nu = \nu(\mathbf{c})$ such that*

$$\sup_{0\leqslant t\leqslant T} \{\|\tau(t)\|^2_{H^2(\mathbb{T}^1)} + \|v(t)\|^2_{H^2(\mathbb{T}^1)}\} + \nu \int_0^T \|\partial_m v(t)\|^2_{H^2(\mathbb{T}^1)} dt \leqslant C. \tag{3.51}$$

Proof. We know that in mass-Lagrangian coordinate, the Navier Stokes equation is written

$$\begin{cases} \partial_t \tau - \partial_m v = 0,\\ \partial_t v + \partial_m \left(\widetilde{P}(\tau) - \widetilde{\mu}(\tau)\partial_m v\right) = 0. \end{cases} \tag{3.52}$$

Multiply $(3.52)_1$ by τ, $(3.52)_2$ by v, summing and integrating on the $\mathbb{T}^1$, for $\varepsilon > 0$, one has :

$$\frac{d}{dt}\{\|\tau(t)\|^2_{L^2(\mathbb{T}^1)} + \|v(t)\|^2_{L^2(\mathbb{T}^1)}\} + \int_{\mathbb{T}^1} (\widetilde{\mu}(\tau) - \varepsilon)|\partial_m v|^2 \leqslant C(\varepsilon, \mathbf{c})\|\tau(t)\|^2_{L^2(\mathbb{T}^1)}. \tag{3.53}$$

Next, multiply $(3.52)_1$ by $\partial_{mm}\tau$, $(3.52)_2$ by $\partial_{mm} v$, summing and integrating on the $\mathbb{T}^1$, one has :

$$\frac{d}{dt}\{\|\partial_m\tau(t)\|^2_{L^2(\mathbb{T}^1)} + \|\partial_m v(t)\|^2_{L^2(\mathbb{T}^1)}\} + \int_{\mathbb{T}^1} (\widetilde{\mu}(\tau) - \varepsilon)|\partial_{mm} v|^2 \leqslant C(\varepsilon, \mathbf{c})\|\partial_m\tau(t)\|^2_{L^2(\mathbb{T}^1)}. \tag{3.54}$$

Taking two derivatives of $(3.52)_1$ with respect to m and multiplying by $\partial_{mm}\tau$ and integrating on the torus, one obtains :

$$\frac{1}{2}\frac{d}{dt}\|\partial_{mm}\tau(t)\|^2_{L^2(\mathbb{T}^1)} \leqslant \varepsilon\|\partial_{mmm} v(t)\|^2_{L^2(\mathbb{T}^1)} + \frac{1}{4\varepsilon}\|\partial_{mm}\tau(t)\|^2_{L^2(\mathbb{T}^1)}. \tag{3.55}$$

Taking two derivatives of $(3.52)_2$ with respect to m, next multiplying by $\partial_{mm} v$ and integrating on the torus, one obtains :

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_{mm} v(t)\|^2_{L^2(\mathbb{T}^1)} + \int_{\mathbb{T}^1} \widetilde{\mu}(\tau)|\partial_{mmm} v|^2 = \int_{\mathbb{T}^1} \partial_{mmm} v \Big[\widetilde{P}''(\tau)|\partial_m\tau|^2 + \widetilde{P}'(\tau)\partial_{mm}\tau\\ -\widetilde{\mu}''(\tau)|\partial_m\tau|^2\partial_m v - \widetilde{\mu}'(\tau)\partial_{mm}\tau\partial_m v - 2\widetilde{\mu}'(\tau)\partial_m\tau\partial_{mm} v\Big].\end{aligned} \tag{3.56}$$

Now we have to estimate each terms appearing in the right hand side of (3.56). By Hölder inequality and Gagliardo-Niremberg one has successively,

$$\begin{aligned}\left|\int_{\mathbb{T}^1} \widetilde{P}''(\tau)\partial_{mmm}v|\partial_m\tau|^2\right| &\leqslant \varepsilon\|\partial_{mmm}v\|^2_{L^2(\mathbb{T}^1)} + C_1(\varepsilon,\mathbf{c})\|\partial_m\tau\|^4_{L^4(\mathbb{T}^1)}\\ &\leqslant \varepsilon\|\partial_{mmm}v\|^2_{L^2(\mathbb{T}^1)} + C_1(\varepsilon,\mathbf{c})\|\partial_{mm}\tau\|_{L^2(\mathbb{T}^1)}\|\partial_m\tau\|^3_{L^2(\mathbb{T}^1)}\\ &\leqslant \varepsilon\|\partial_{mmm}v\|^2_{L^2(\mathbb{T}^1)} + C_1(\varepsilon,\mathbf{c})\left(\|\partial_{mm}\tau\|^2_{L^2(\mathbb{T}^1)} + \|\partial_m\tau\|^6_{L^2(\mathbb{T}^1)}\right);\end{aligned} \tag{3.57}$$

$$\left|\int_{\mathbb{T}^1} \widetilde{P}'(\tau)\partial_{mmm}v\partial_{mm}\tau\right| \leqslant \varepsilon\|\partial_{mmm}v\|^2_{L^2(\mathbb{T}^1)} + C_3(\varepsilon,\mathbf{c})\|\partial_{mm}\tau\|^2_{L^2(\mathbb{T}^1)}; \tag{3.58}$$

$$\begin{aligned}\left|\int_{\mathbb{T}^1} \widetilde{\mu}''(\tau)\partial_{mmm}v|\partial_m\tau|^2\partial_m v\right| &\leqslant C_4(\mathbf{c})\|\partial_{mmm}v\|_{L^2(\mathbb{T}^1)}\|\partial_m v\|_{L^2(\mathbb{T}^1)}\|\partial_m\tau\|^2_{L^\infty(\mathbb{T}^1)}\\ &\leqslant 2C_4(\mathbf{c})\|\partial_{mmm}v\|_{L^2(\mathbb{T}^1)}\|\partial_m v\|_{L^2(\mathbb{T}^1)}\|\partial_{mm}\tau\|_{L^2(\mathbb{T}^1)}\|\partial_m\tau\|_{L^2(\mathbb{T}^1)}\\ &\leqslant \varepsilon\|\partial_{mmm}v\|^2_{L^2(\mathbb{T}^1)} + \frac{C_4(\mathbf{c})}{2\varepsilon}\|\partial_{mm}\tau\|^2_{L^2(\mathbb{T}^1)}\|\partial_m\tau\|^2_{L^2(\mathbb{T}^1)}\|\partial_m v\|^2_{L^2(\mathbb{T}^1)};\end{aligned} \tag{3.59}$$

$$\left|\int_{\mathbb{T}^1} \widetilde{\mu}'(\tau)\partial_{mmm}v\partial_{mm}\tau\partial_m v\right| \leqslant \varepsilon\|\partial_{mmm}v\|^2_{L^2(\mathbb{T}^1)} + C_5(\varepsilon,\mathbf{c})\|\partial_{mm}\tau\|^2_{L^2(\mathbb{T}^1)}\|\partial_m v\|^2_{L^\infty(\mathbb{T}^1)}; \tag{3.60}$$

$$\begin{aligned}\left|\int_{\mathbb{T}^1} \widetilde{\mu}'(\tau)\partial_{mmm}v\partial_m\tau\partial_{mm}v\right| &\leqslant C_6(\mathbf{c})\|\partial_{mmm}v\|_{L^2(\mathbb{T}^1)}\|\partial_{mm}v\|_{L^\infty(\mathbb{T}^1)}\|\partial_m\tau\|_{L^2(\mathbb{T}^1)}\\ &\leqslant \sqrt{2}C_6(\mathbf{c})\|\partial_{mmm}v\|^{3/2}_{L^2(\mathbb{T}^1)}\|\partial_{mm}v\|^{1/2}_{L^2(\mathbb{T}^1)}\|\partial_m\tau\|_{L^2(\mathbb{T}^1)}\\ &\leqslant \varepsilon\|\partial_{mmm}v\|^2_{L^2(\mathbb{T}^1)} + \frac{\sqrt{2}C_6(\mathbf{c})}{4\varepsilon}\|\partial_{mm}v\|^2_{L^2(\mathbb{T}^1)}\|\partial_m\tau\|^4_{L^2(\mathbb{T}^1)}.\end{aligned} \tag{3.61}$$

First combining (3.53) and (3.54), and applying Grönwall lemma, one has :

$$\sup_{0\leqslant t\leqslant T}\{\|\tau(t)\|^2_{H^1(\mathbb{T}^1)} + \|v(t)\|^2_{H^1(\mathbb{T}^1)}\} + \nu\int_0^T \|\partial_m v(t)\|^2_{H^1(\mathbb{T}^1)}dt \leqslant C_7. \tag{3.62}$$

Next, combining (3.55), (3.56), (3.57), (3.58), (3.59), (3.60) and (3.61), one finds a constant $C_8(\varepsilon,\mathbf{c})$ such that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\|\partial_{mm}\tau(t)\|^2_{L^2(\mathbb{T}^1)} + \|\partial_{mm}v(t)\|^2_{L^2(\mathbb{T}^1)}\right] + \int_{\mathbb{T}^1}(\widetilde{\mu}(\tau) - 6\varepsilon)|\partial_{mmm}v|^2 \leqslant C_8(\varepsilon,\mathbf{c})\Big[\|\partial_m\tau\|^6_{L^2(\mathbb{T}^1)} +\\ &\left(\|\partial_{mm}\tau\|^2_{L^2(\mathbb{T}^1)} + \|\partial_{mm}v\|^2_{L^2(\mathbb{T}^1)}\right)\left(1 + \|\partial_m\tau\|^2_{L^2(\mathbb{T}^1)}\|\partial_m v\|^2_{L^2(\mathbb{T}^1)} + \|\partial_m v\|^2_{L^\infty(\mathbb{T}^1)} + \|\partial_m\tau\|^4_{L^2(\mathbb{T}^1)}\right)\Big].\end{aligned} \tag{3.63}$$

To apply the Grönwall to the above equality, we need $\partial_m\tau \in L^6((0,T),L^2(\mathbb{T}^1))$, $\partial_m v \in L^4((0,T),L^2(\mathbb{T}^1))$ and $\partial_m v \in L^2((0,T),L^\infty(\mathbb{T}^1))$. The two first part of these assumptions are ensured by (3.62). The last one is the motivation of Lemma 3.3.2. In consequence, thanks to Lemma 3.3.2 and (3.62), one easily applies the Grönwall lemma and obtains the lemma. □

And the Theorem 3.3.2 follows. □

CHAPTER 4

EXISTENCE OF CLASSICAL AND GLOBAL SOLUTION

4.1 Main results

In the previous chapter, we showed that if the density satisfies (3.44), the solution that we built, is globally defined in time. It is natural to want to know under what condition on the initial data, (3.44) is satisfied. In [2], Bresch, Perrin and Zatorska address this problem by using the so-called Bresch-Desjardins entropy with singular pressure on the form :

$$P(\rho) = \frac{\rho^\gamma}{(1-\rho)^\beta}, \quad \beta, \gamma > 1.$$

One observes that the Navier Stokes system can be written as follows :

$$\begin{cases} \partial_t \widetilde{\rho} + \widetilde{\rho}^2 \partial_m v = 0, \\ \partial_t v + \partial_m (P(\widetilde{\rho}) - \mu(\widetilde{\rho}) \widetilde{\rho} \partial_m v) = 0. \end{cases} \tag{4.1}$$

Then multiply the second equation of the above system by the so-called effective velocity

$$v + \frac{\mu(\widetilde{\rho})}{\widetilde{\rho}} \partial_m \widetilde{\rho}$$

and integrate on $\mathbb{T}^1$, one has :

$$\frac{d}{dt} \int_{\mathbb{T}^1} \left(\frac{1}{2} \left| v + \frac{\mu(\widetilde{\rho})}{\widetilde{\rho}} \partial_m \widetilde{\rho} \right|^2 + e(\rho) \right) + \int_{\mathbb{T}^1} P'(\rho) \rho^{-1} |\partial_m \rho|^2 = 0.$$

If we assume that the viscosity equals 1, one has a $L^\infty_{\text{loc}}(\mathbb{R}^+, L^2(\mathbb{T}^1))$ bound on the effective velocity provided that $v_0 + \partial_m \ln(\widetilde{\rho}_0)$ belongs to $L^2(\mathbb{T}^1)$ and therefore $\partial_m \ln(\widetilde{\rho}_0) = \widetilde{\rho} \partial_x \ln(\rho_0) \in L^2(\mathbb{T}^1)$ and consequently $\partial_x \rho_0 \in L^2(\mathbb{T}^1)$, because, it is natural to assume that the initial velocity belongs to $L^2(\mathbb{T}^1)$ when working with the Navier Stokes system with finite energy. This shows that they use the H^1 assumption on the initial density in their computations.

In the following, we obtain the same result only assuming finite (small) initial energy and as a consequence, one obtains the existence of global classical solution of the Navier Stokes system.

The proof is inspired by the Lemma 1.3 of [25]. Throughout this section, the singular pressure take the form

$$P(\rho) = \frac{\rho^\gamma}{(1-\rho)^\beta}, \quad \beta, \gamma > 1. \tag{4.2}$$

The main result of this chapter is the following :

Theorem 4.1.1. *Assume that there exists $\alpha_0 > 0$ such that*

$$0 < \alpha_0 \leqslant \rho_0 \leqslant 1 - \alpha_0.$$

Then there exist $0 < \alpha < \alpha_0$ independent of E_0, a non negative constant $C = C(E_0)$ such that for all $T > 0$, there exists $c = c(T, E_0) > 0$ such that

$$\forall\, 0 \leqslant t \leqslant T, \quad 0 < c \leqslant \widetilde{\rho}(t) \leqslant (1-\alpha)\exp C.$$

In particular, if $E_0 << 1$, then there exists $0 < \widetilde{c} = \widetilde{c}(E_0) < 1$ such that :

$$\forall\, 0 \leqslant t \leqslant T, \quad 0 < c \leqslant \widetilde{\rho}(t, m) \leqslant \widetilde{c}.$$

Gathering the above theorem, the existence theorem Theorem 3.1.1, the uniqueness theorem Theorem 3.2.1 and the blow-up criterion Theorem 3.3.2, one has the following existence and uniqueness of classical and global solution of the Cauchy problem associated to the Navier Stokes system.

Theorem 4.1.2. *Assume that $(\rho_0, u_0) \in H^2(\mathbb{T}^1) \times H^2(\mathbb{T}^1)$, the pressure P is like (4.2) and that there exists $\alpha_0 > 0$ such that*

$$\alpha_0 \leqslant \rho_0 \leqslant \rho_{max} - \alpha_0$$

and the initial energy $E_0 << 1$. Then, there exists a unique classical solution

$$(\rho, u) \in \mathscr{C}(\mathbb{R}^+, H^2(\mathbb{T}^1)) \times \big(\mathscr{C}(\mathbb{R}^+, H^2(\mathbb{T}^1)) \cap L^2_{loc}(\mathbb{R}^+, H^3(\mathbb{T}^1))\big)$$

of the Cauchy problem

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P(\rho) = \partial_{xx} u, \\ \rho(0) = \rho_0, \quad u(0) = u_0. \end{cases}$$

Moreover, there exists $\widetilde{c} = \widetilde{c}(E_0) < 1$ such that for all $T > 0$, there exists $0 < c = c(T, E_0)$ such

$$0 < c \leqslant \rho \leqslant \widetilde{c} < 1,$$

and for any $T > 0$, there exists $C = C(T, \alpha_0, \|\rho_0\|_{H^2(\mathbb{T}^1)}, \|u_0\|_{H^2(\mathbb{T}^1)}) > 0$ such that :

$$\sup_{0 \leqslant t \leqslant T} \{\|\rho(t)\|^2_{H^2(\mathbb{T}^1)} + \|u(t)\|^2_{H^2(\mathbb{T}^1)}\} + \int_0^T |\partial_x u|^2_{H^2(\mathbb{T}^1)}\} \leqslant C.$$

The proof of Theorem 4.1.1 follows after the following results.

Let us define e and the energy E respectively by :

$$e(\rho) = \int_{\overline{\rho}}^{\rho} s^{-2}\left[P(s) - P(\overline{\rho})\right] ds \text{ and } E(t) = \int_{\mathbb{T}^1} \left[|v(t,m)|^2 + e(\widetilde{\rho}(t,m))\right] dm$$

where $\overline{\rho} = \dfrac{1}{|\mathbb{T}^1|}\displaystyle\int_{\mathbb{T}^1} \rho_0(x)dx$. The classical energy estimate of the Navier Stokes system leads to :

Proposition 4.1.1. *Assume that* $(\rho_0, u_0) \in \left(H^2(\mathbb{T}^1)\right)^2$. *Then,*

$$\sup_{0\leqslant t\leqslant T} E(t) + \int_0^T \int_{\mathbb{T}^1} \widetilde{\rho}|\partial_m v|^2(t,m)dtdm \leqslant E_0$$

with

$$E_0 = \int_{\mathbb{T}^1} \left[|v_0(m)|^2 + e(\widetilde{\rho}_0(m))\right] dm < \infty.$$

Let us define φ by

$$\varphi(t,m) = \frac{1}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\int_q^m \left[\frac{1}{\widetilde{\rho}(t,q')} - \frac{1}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\frac{dz}{\widetilde{\rho}(t,z)}\right] dq'dq. \tag{4.3}$$

The function φ is a periodic function differential so that we can use it as test function in the momentum equation. The exceptional feature of φ is that its space derivative can be bound by below when the density is closer to 1.

By testing φ with the momentum equation, one has the following :

Lemma 4.1.1.

$$\begin{aligned}\left|\int_0^T \int_{\mathbb{T}^1} P(\widetilde{\rho})\left[\frac{1}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\frac{dz}{\widetilde{\rho}_0(z)} - \frac{1}{\widetilde{\rho}}\right](t,m)dtdm\right|&\\ \leqslant 2E_0^{1/2}M_0^{1/2} + \sqrt{T}E_0^{1/2}\left[|\mathbb{T}^1|^{1/2}E_0^{1/2} + M_0^{1/2}\left[1+|\mathbb{T}^1|^{-1/2}M_0^{1/2}\right]\right]&\end{aligned} \tag{4.4}$$

with

$$M_0 = \int_{\mathbb{T}^1}\frac{dm}{\widetilde{\rho}_0(m)}.$$

Because of the singularity of the pressure it is not sure that $P(\widetilde{\rho})$ is integrable. But, the above lemma leads to the following :

Lemma 4.1.2. *One has* $P(\widetilde{\rho})$ *and* $\widetilde{\rho}^{-1}P(\widetilde{\rho})$ *are* $L^1((0,T)\times\mathbb{T}^1)$.

From this lemma, one deduces a lower and up bound for the density when the initial energy is small. The result is stated in the following :

Lemma 4.1.3. *There exist* $0 < \alpha < \alpha_0$ *independent of* E_0 *and a constant* $C = C(E_0)$ *going to zero when* E_0 *tends to zero, such that*

$$\widetilde{\rho} \leqslant (1-\alpha)\exp C.$$

In particular, for small E_0, *one can bound the density far from the singularity of the pressure.*

Lemma 4.1.4. *There exists a constant* $C = C(T, E_0)$ *such that*

$$\forall\, 0 \leqslant t \leqslant T, \quad \widetilde{\rho}(t)) \geqslant C.$$

4.2 Proofs

Proof of Lemma 4.1.1. Testing φ defined in (4.3) with the momentum equation, one obtains

$$\int_{(\mathbb{T}^1)} v(T,m)\varphi(T,m) - \int_{\mathbb{T}^1} v(0,m)\varphi(0,m) = \int_0^T\int_{\mathbb{T}^1} v(t,m)\partial_t\varphi(t,m)dtdm + \int_0^T\int_{\mathbb{T}^1} \left(P(\widetilde{\rho}) - \widetilde{\rho}\partial_m v(t,m)\right)\partial_m\varphi(t,m)dtdm. \tag{4.5}$$

Noting that

$$\partial_t\varphi(t,m) = v(t,m) - \frac{1}{|\mathbb{T}^1|}\int_{\mathbb{T}^1} v(t,q)dq = \frac{1}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\int_q^m \partial_{q'}v(t,q')dq'dq$$

and

$$\partial_m\varphi(t,m) = \frac{1}{\widetilde{\rho}(t,m)} - \frac{1}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\frac{dq}{\widetilde{\rho}(t,q)},$$

then (4.5) implies

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^1} P(\widetilde{\rho})\left[\frac{1}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\frac{dz}{\widetilde{\rho}_0(z)} - \frac{1}{\widetilde{\rho}}\right](t,m)dtdm &= \int_{\mathbb{T}^1} v(0,m)\varphi(0,m) - \int_{\mathbb{T}^1} v(T,m)\varphi(T,m)\\ &+ \frac{1}{|\mathbb{T}^1|}\int_0^T\int_{\mathbb{T}^1} v(t,m)\int_{\mathbb{T}^1}\int_q^m \partial_{q'}v(t,q')dq'dqdtdm\\ &- \int_0^T\int_{\mathbb{T}^1}\widetilde{\rho}\partial_m v(t,m)\left[\frac{1}{\widetilde{\rho}(t,m)} - \frac{1}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\frac{dq}{\widetilde{\rho}(t,q)}\right].\end{aligned} \tag{4.6}$$

In the following we estimate terms appearing in the right hand side of the above equality. We begin by the third term.

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^1} v(t,m)\int_{\mathbb{T}^1}\int_q^m \partial_{q'}v(t,q')dq'dqdmdt &\leqslant \int_0^T\left[\int_{\mathbb{T}^1}|v(t,m)|dm\right]\left[\int_{\mathbb{T}^1}|\partial_q v(t,q)|dq\right]dt\\ &\leqslant \left[\int_0^T\left[\int_{\mathbb{T}^1}|v(t,m)|dm\right]^2\right]^{1/2}\left[\int_0^T\left[\int_{\mathbb{T}^1}|\partial_q v(t,q)|dq\right]^2\right]^{1/2}\\ &\leqslant |\mathbb{T}^1|\left[\int_0^T\int_{\mathbb{T}^1}|v(t,m)|^2dm\right]^{1/2}\left[\int_0^T\int_{\mathbb{T}^1}|\partial_q v(t,q)|^2dq\right]^{1/2}\\ &\leqslant |\mathbb{T}^1|\sqrt{T}\left[\sup_{0\leqslant t\leqslant T}\int_{\mathbb{T}^1}|v(t,m)|^2dm\right]^{1/2}\left[\int_0^T\int_{\mathbb{T}^1}|\partial_q v(t,q)|^2dq\right]^{1/2}.\end{aligned}$$

Then, using estimation in Proposition 4.1.1, one obtains :

$$\left|\int_0^T\int_{\mathbb{T}^1} v(t,m)\int_{\mathbb{T}^1}\int_q^m \partial_{q'}v(t,q')\right| \leqslant |\mathbb{T}^1|\sqrt{T}E_0. \tag{4.7}$$

The last terms

$$\left|\int_0^T\int_{\mathbb{T}^1}\widetilde{\rho}\partial_m v\left[\frac{1}{\widetilde{\rho}} - \frac{1}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\frac{dq}{\widetilde{\rho}(t,q)}\right]\right| \leqslant \sqrt{T}E_0^{1/2}\left[\int_{\mathbb{T}^1}\frac{dm}{\widetilde{\rho}_0(m)}\right]^{1/2}\left[1 + |\mathbb{T}^1|^{-1/2}\left[\int_{\mathbb{T}^1}\frac{dm}{\widetilde{\rho}_0(m)}\right]^{1/2}\right]. \tag{4.8}$$

As well

$$\left|\int_{\mathbb{T}^1} v(0,m)\varphi(0,m) - \int_{(\mathbb{T}^1)} v(T,m)\varphi(T,m)\right| \leqslant 2E_0^{1/2}\left[\int_{\mathbb{T}^1}\frac{dm}{\widetilde{\rho}_0(m)}\right]^{1/2}. \tag{4.9}$$

Then, combining (4.6), (4.7), (4.8) and (4.9) one obtains (4.4). □

Proof of Lemma 4.1.2. The proof consists in separating the integral of (4.4) into two parts : where the density is closer to 1 and its complementary.

For $\eta > 0$ let us define $\Gamma(t) = \{m \in \mathbb{T}^1 \colon \widetilde{\rho}(t,m) \geqslant \eta\}$ and set

$$\frac{1}{\eta} = \frac{1-\varepsilon}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\frac{dq}{\widetilde{\rho}_0(q)}.$$

On $\Gamma(t)$,

$$\frac{1}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\frac{dq}{\widetilde{\rho}_0(q)} - \frac{1}{\widetilde{\rho}} \geqslant \frac{\varepsilon}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\frac{dq}{\widetilde{\rho}_0(q)}.$$

With the uniform boundness of the initial density, one can choose a very small $\varepsilon > 0$ such

$$\eta = (1-\varepsilon)^{-1}|\mathbb{T}^1|\left[\int_{\mathbb{T}^1}\frac{dq}{\widetilde{\rho}_0(q)}\right]^{-1} < 1,$$

for example

$$\varepsilon < 1 - |\mathbb{T}^1|\left[\int_{\mathbb{T}^1}\frac{dq}{\widetilde{\rho}_0(q)}\right]^{-1}.$$

This is possible provided that the initial density is not identically 1.

On $\Gamma^c(t)$, there is no singularity and

$$\frac{1}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\frac{dq}{\widetilde{\rho}_0(q)} - \frac{1}{\widetilde{\rho}} \leqslant \frac{\varepsilon}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\frac{dq}{\widetilde{\rho}_0(q)}$$

then one has, with increasing pressure

$$\int_0^T\int_{\Gamma^c(t)} P(\widetilde{\rho})\left[\frac{1}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\frac{dz}{\widetilde{\rho}_0(z)} - \frac{1}{\widetilde{\rho}}\right](t,m)dtdm \leqslant \frac{\varepsilon T P(\eta)}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\frac{dq}{\widetilde{\rho}_0(q)}$$

thus

$$\begin{aligned}\int_0^T\int_{\Gamma(t)} P(\widetilde{\rho})\left[\frac{1}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\frac{dz}{\widetilde{\rho}_0(z)} - \frac{1}{\widetilde{\rho}}\right](t,m)dtdm \leqslant \int_0^T\int_{\mathbb{T}^1} P(\widetilde{\rho})\left[\frac{1}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\frac{dz}{\widetilde{\rho}_0(z)} - \frac{1}{\widetilde{\rho}}\right](t,m)dtdm \\ + \varepsilon T P(\eta)\int_{\mathbb{T}^1}\frac{dq}{\widetilde{\rho}_0(q)}.\end{aligned}$$

As well

$$\int_0^T\int_{\Gamma(t)} P(\widetilde{\rho}) \leqslant \left[\varepsilon\frac{1}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\frac{dq}{\widetilde{\rho}(q)}\right]^{-1}\left[\int_0^T\int_{\mathbb{T}^1} P(\widetilde{\rho})\left[\frac{1}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\frac{dz}{\widetilde{\rho}_0(z)} - \frac{1}{\widetilde{\rho}}\right] + \frac{\varepsilon T P(\eta)}{|\mathbb{T}^1|}\int_{\mathbb{T}^1}\frac{dq}{\widetilde{\rho}_0(q)}\right]$$

And consequently, $P(\widetilde{\rho})$ and $\widetilde{\rho}^{-1}P(\widetilde{\rho})$ are $L^1((0,T)\times\mathbb{T}^1)$.

Moreover, using Lemma 4.1.1, there exist two constants C_1, and C_2 depending only on E_0, with C_1 tending to 0 when E_0 tends to 0, such that

$$\|P(\widetilde{\rho})\|_{L^1((0,T)\times\mathbb{T}^1)} + \|\widetilde{\rho}^{-1}P(\widetilde{\rho})\|_{L^1((0,T)\times\mathbb{T}^1)} \leqslant C_1 + C_2 T. \tag{4.10}$$

Remark 4.2.1. *Let us remark that C_2 is given by :*

$$C_2 = P(\eta) + \frac{|\mathbb{T}^1|}{2\varepsilon} M_0^{-1} \left[|\mathbb{T}^1|^{1/2} E_0^{1/2} + M_0^{1/2} \left[1 + |\mathbb{T}^1|^{-1/2} M_0^{1/2} \right] \right]^2$$

and

$$C_1 = \frac{1}{\varepsilon} M^{-1} \left[2E_0^{1/2} M_0^{1/2} + \frac{1}{2} E_0 \right]$$

and η and ε do not depend on E_0 but only on α_0.

Now, we turn back to the proof of Lemma 4.1.3.

Proof of Lemma 4.1.3. Let us rewrite the Navier Stokes system in mass-Lagrangian coordinate

$$\begin{cases} \partial_t \widetilde{\rho} + \widetilde{\rho}^2 \partial_m v = 0, \\ \partial_t v + \partial_m (P(\widetilde{\rho}) - \widetilde{\rho} \partial_m v) = 0. \end{cases} \tag{4.11}$$

Multiplying $(4.11)_1$ by $\widetilde{\rho}^{-1}$, one obtains

$$\partial_t \psi(\tau) = -\widetilde{\rho} \partial_m v$$

where ψ is the logarithm function. Then $(4.11)_2$ becomes

$$\partial_t \left(v + \partial_m \psi(\widetilde{\rho}) \right) + \partial_m P(\widetilde{\rho}) = 0. \tag{4.12}$$

Integrating the above identity on (l, q) with respect to m, and on the $\mathbb{T}^1$ with respect to l, one has :

$$\partial_t \psi(\widetilde{\rho}(t,q)) = -P(\widetilde{\rho}(t,q)) - \frac{d}{dt} \int_{\mathbb{T}^1} \int_l^q v(t,m) dm dl + \int_{\mathbb{T}^1} P(\widetilde{\rho}(t,l)) dl - \int_{\mathbb{T}^1} \widetilde{\rho} \partial_m v(t,l) dl. \tag{4.13}$$

Let us observe that, after the integration over (l, q), one replaces $\partial_t \psi(t,l)$ by $-\widetilde{\rho}\partial_m v(t,l)$ before integrating on the torus with respect to l. Now, we fix $q \in \mathbb{T}^1$. For $0 < \alpha < \alpha_0$, we know that the set

$$A_\alpha(q) := \{ t \in [0,T] \colon \forall 0 \leqslant s \leqslant t,\ \widetilde{\rho}(s,q) \leqslant 1 - \alpha \}$$

is not empty since $0 \in A_\alpha(q)$. Let $t_\alpha(q)$ be the upper bound of $A_\alpha(q)$. As we don't know the monotony of $\widetilde{\rho}(t, \cdot)$, namely the sign of $\partial_m v(t, \cdot)$, we are unable to prove that $\forall t > t_\alpha$, $\widetilde{\rho}(t,q) > 1-\alpha$. But, thanks to the continuity of $\widetilde{\rho}$, one can write $\{t > t_\alpha(q) \colon \widetilde{\rho}(t,q) > 1 - \alpha\}$ as union of intervals. Let (s,t) an interval of the union, for any $t' \in (s,t)$, one has $\widetilde{\rho}(t',q) > 1 - \alpha$ then

$$-P(\widetilde{\rho}(t',q)) \leqslant -\frac{(1-\alpha)^\gamma}{\alpha^\beta}. \tag{4.14}$$

Integrating (4.13) between s and t' one obtains :

$$\psi(\widetilde{\rho}(t',q)) - \psi(\widetilde{\rho}(s,q)) = -\int_s^{t'} P(\widetilde{\rho}(\tau,q)) d\tau + b(t') - b(s) \tag{4.15}$$

with

$$b(t) = \int_0^t \int_{\mathbb{T}^1} P(\widetilde{\rho}(s,l)) ds dl - \int_0^t \int_{\mathbb{T}^1} \widetilde{\rho} \partial_m v(s,l) ds dl - \int_{\mathbb{T}^1} \int_l^q v(t,m) dm dl.$$

By following what we did to get (4.10), one obtains :

$$\int_s^{t'}\int_{\mathbb{T}^1} P(\widetilde{\rho}(\tau,l))d\tau dl \leqslant C_1 + C_2(t'-s).$$

As well, using the energy estimates, ones has

$$\int_s^{t'}\int_{\mathbb{T}^1} \widetilde{\rho}\partial_m v(\tau,l)d\tau dl \leqslant E_0^{1/2}\sqrt{t'-s} \leqslant \frac{E_0}{2}+\frac{1}{2}(t'-s)$$

and

$$\int_{\mathbb{T}^1}\int_l^q (v(t',m)-v(s,m))dmdl \leqslant 2E_0^{1/2}.$$

Combining the three above inequalities, one has :

$$b(t')-b(s) \leqslant C(E_0) + (C_2+\frac{1}{2})(t'-s) \tag{4.16}$$

with a constant C depending on E_0 and goes to zero when E_0 tends to zero.

Let us take α small such that

$$C_2 + \frac{1}{2} \leqslant \frac{(1-\alpha)^\gamma}{\alpha^\beta}$$

and combine (4.14), (4.15) and (4.16), to obtain :

$$\psi(\widetilde{\rho}(t',q)) - \psi(\widetilde{\rho}(s,q)) \leqslant C(E_0).$$

Then, noting that, by continuity of the density, $\widetilde{\rho}(s,q) = 1-\alpha$, one gets other $\{t > t_\alpha \colon \widetilde{\rho}(t,q) > 1-\alpha\}$

$$\psi(\widetilde{\rho}(t',q)) \leqslant C(E_0) + \psi(1-\alpha).$$

Since $C(E_0)$ neither α does not depend on q , we conclude that the density satisfies the bound

$$\psi(\widetilde{\rho}) \leqslant C(E_0) + \psi(1-\alpha).$$

Let us recall that ψ is the logarithm function, so we obtain :

$$\widetilde{\rho} \leqslant (1-\alpha)\exp\left(C(E_0)\right).$$

By the Remark 4.2.1, when E_0 tends to zero, C_2 does not tend to infinity but decreases and consequently α do not tend to zero so can be bound by below. □

Proof of Lemma 4.1.4. The momentum equation in mass-Lagrangian coordinate is written:

$$\partial_t\left(v + \partial_m\psi(\widetilde{\rho})\right) = -\partial_m P(\widetilde{\rho}). \tag{4.17}$$

Integrating (4.17) on (q,q') with respect to m and then integrating the obtained equation on the $\mathbb{T}^1$ with respect to q', and finally on (0,t) in time, one has,

$$\begin{aligned}\ln\left(\widetilde{\rho}(t,q)\right) = &\int_{\mathbb{T}^1}\int_{q'}^q v_0(m)dm - \int_{\mathbb{T}^1}\int_{q'}^q v(t,m)dm + \int_{\mathbb{T}^1}\ln\left(\widetilde{\rho}(t,q')\right)dq' + \ln\left(\widetilde{\rho}_0(q)\right)\\ &- \int_{\mathbb{T}^1}\ln\left(\widetilde{\rho}_0(q')\right)dq' + \int_0^t\int_{\mathbb{T}^1} P\left(\widetilde{\rho}(s,q')\right)dq'ds - \int_0^t P\left(\widetilde{\rho}(s,q)\right)ds.\end{aligned} \tag{4.18}$$

From mass conservation equation in mass-Lagrangian coordinate, we have

$$\frac{d}{dt}\int_{\mathbb{T}^1} \ln\left(\widetilde{\rho}(t,m)\right) dm = \int_{\mathbb{T}^1} \widetilde{\rho}(t,m)\partial_m v(t,m) dm \leqslant \left[\int_{\mathbb{T}^1} \widetilde{\rho}|\partial_m v|^2 dm\right]^{1/2}.$$

Then integrating the above inequality on $(0,t)$ and combining with (4.18), one has

$$\begin{aligned} \ln\left(\widetilde{\rho}(t,q)\right) \leqslant \int_{\mathbb{T}^1}\int_{q'}^{q} v_0(m) dm - \int_{\mathbb{T}^1}\int_{q'}^{q} v(t,m) dm + \sqrt{T}\left[\int_0^T \int_{\mathbb{T}^1} \widetilde{\rho}|\partial_m v|^2 dm\right]^{1/2} \\ + \ln\left(\widetilde{\rho}_0(q)\right) + \int_0^t \int_{\mathbb{T}^1} P\left(\widetilde{\rho}(s,q')\right) dq' ds - \int_0^t P\left(\widetilde{\rho}(s,q)\right) ds. \end{aligned} \tag{4.19}$$

Each terms appearing in the right hand side of the above equation can be bounded by a constant depending only on the initial data and T. This ends the proof. □

CHAPTER 5

WEAK SOLUTIONS *À LA HOFF*

This chapter is devoted to constructing a distributional solution of the Navier Stokes system with singular pressure only assuming that the initial energy is small. We use techniques introduced by Hoff in [12, 13] to obtain for example that the space derivative of the velocity $\partial_x u$ belongs to $L^1_{\text{loc}}(\mathbb{R}^+, L^\infty(\mathbb{T}^1))$. This enables us to formally define the flow associated to the velocity u. In the sequel, the viscosity is constant taken equal 1 and the pressure is :

$$P(\rho) = \frac{\rho^\gamma}{(1-\rho)^\beta}. \tag{5.1}$$

We mainly exploit [4] for Hoff estimates.

5.1 Main result

The existence result of weak solutions *à la Hoff* is stated in the following theorem.

Theorem 5.1.1. *Let $u_0 \in L^2(\mathbb{T}^1)$, $\rho_0 \in L^\infty(\mathbb{T}^1)$ and assume that there exists $\alpha_0 > 0$ such that*

$$0 < \alpha_0 \leqslant \rho \leqslant 1 - \alpha_0.$$

Then there exists a non negative constant c such that if the initial energy

$$E_0 := \int_{\mathbb{T}^1} \rho_0 \left(|u_0|^2 + e(\rho_0)\right) \ \text{with} \ \ e(\rho_0) = \int_{\overline{\rho_0}}^{\rho_0} \frac{P(s) - P(\overline{\rho_0})}{s^2} ds \ \text{and} \ \overline{\rho_0} = \frac{1}{|\mathbb{T}^1|} \int_{\mathbb{T}^1} \rho_0(x) dx$$

satisfies $E_0 \leqslant c$, then there exists

$$(\rho, u) \in L^\infty(\mathbb{R}^+, L^\infty(\mathbb{T}^1)) \times \left(L^2(\mathbb{R}^+, H^1(\mathbb{T}^1)) \cap L^\infty(\mathbb{R}^+, L^2(\mathbb{T}^1))\right)$$

distributional solution of the Navier Stokes system :

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P(\rho) = \partial_{xx} u, \\ \rho(0) = \rho_0, \quad u(0) = u_0. \end{cases} \tag{5.2}$$

Moreover,

$$\sup_t \int_{\mathbb{T}^1} \rho \frac{|u|^2}{2} + \int_0^\infty \int_{\mathbb{T}^1} |\partial_x u|^2 \leqslant E_0 \tag{5.3}$$

and for all $T > 0$ there exists a constant $C(T, E_0)$ such that :

$$\int_0^T \int_{\mathbb{T}^1} \sigma\rho|\dot{u}|^2 + \frac{1}{2} \sup_{0\leqslant t\leqslant T} \sigma(t) \int_{\mathbb{T}^1} |\partial_x u(t)|^2 \leqslant C,$$

$$\sup_{0\leqslant t\leqslant T} \sigma^2(t) \int_{\mathbb{T}^1} \rho|\dot{u}(t)|^2 + \int_0^T \int_{\mathbb{T}^1} \sigma^2 |\partial_x \dot{u}|^2 \leqslant C$$

where $\sigma(t) = \min(1,t)$ and $\dot{u} = \partial_t u + u\partial_x u$ the material derivative.

The proof of this theorem is done by a mollifying method : first, we mollify the initial data (ρ_0, u_0) to obtain a sequence of regular function (ρ_0^n, u_0^n). The smallness of the initial energy guarantee, thanks to Theorem 4.1.2 that the Cauchy problem associated to the Navier Stokes system with the initial data (ρ_0^n, u_0^n) admits, a global classical solution (ρ^n, u^n). Finally we prove that (ρ^n, u^n) admits a sub-sequence that converges weakly to (ρ, u) distributional solution of (5.2). The main difficulty is to identify the weak limit of $(P(\rho^n))_n$. Everything that follows is devoted to the proof of the above theorem.

5.2 Uniform estimates

5.2.1 Mollifying of initial data

Let $\chi\colon \mathbb{T}^1 \to \mathbb{R}^+$ a smooth function satisfying $0 \leqslant \chi \leqslant 1$ and $\displaystyle\int_{\mathbb{T}^1} \chi(t,x)dx = 1$. For all $n \in \mathbb{N}^*$, let us set

$$\chi_n = n\chi(\frac{\cdot}{n}); \quad \rho_0^n = \rho * \chi_n \quad \text{and} \quad u_0^n = u_0 * \chi_n.$$

Using Young inequality and the convolution formula one obtains easily that

$$\forall\, n \in \mathbb{N}^* \quad \alpha_0 \leqslant \rho_0^n \leqslant 1 - \alpha_0.$$

The smallness of E_0 and the smoothness of ρ_0^n and u_0^n ensure that the Cauchy problem

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x P(\rho) = \partial_{xx} u, \\ \rho(0) = \rho_0^n, \quad u(0) = u_0^n \end{cases} \tag{5.4}$$

admits a unique, classical and global solution (ρ^n, u^n) such that $\rho^n \in \mathscr{C}([0,+\infty), H^2(\mathbb{T}^1))$ and $u^n \in \mathscr{C}([0,+\infty), H^2(\mathbb{T}^1)) \cap L^2_{\text{loc}}(\mathbb{R}^+, H^3(\mathbb{T}^1))$. Also, there is a small $0 < \alpha = \alpha(E_0) < \alpha_0$ and for all $T > 0$ there exists $c = c(T) > 0$ such that

$$\forall\, n \quad 0 < c \leqslant \rho^n \leqslant 1 - \alpha. \tag{5.5}$$

These regularities on ρ^n and u^n justify all computations in sections 5.2.2 and 5.2.3.

Remark 5.2.1. *It is important to note that the uniform bound (5.5) of the density ensures that $P(\rho^n)$ and $P'(\rho^n)$ are bounded in $L^\infty((0,T)\times\mathbb{T}^1)$ independently of n because the density is far from the singular point of P, uniformly in n.*

The classical energy estimate leads to the following proposition.

Proposition 5.2.1. *For any $T>0$*

$$\sup_{0\leqslant t\leqslant T} E(\rho^n(t),u^n(t)) + \int_0^T\int_{\mathbb{T}^1}|\partial_x u^n|^2 \leqslant E_0.$$

In what follows, we use techniques introduced by David Hoff in [12, 13] to obtain two additional estimates on (ρ^n,u^n). Let us recall that (ρ^n,u^n) satisfies

$$\begin{cases}\partial_t\rho^n+\partial_x(\rho^n u^n)=0,\\ \partial_t(\rho^n u^n)+\partial_x\left(\rho^n(u^n)^2+P(\rho^n)-\partial_x u^n\right)=0\end{cases} \tag{5.6}$$

and introduce the material derivative $\dot{u}^n=\partial_t u^n+u^n\partial_x u^n$, $\dot{\rho}^n=\partial_t\rho^n+u^n\partial_x\rho^n$ and observe that the above equation can be written as follows :

$$\begin{cases}\dot{\rho}^n=-\rho^n\partial_x u,\\ \rho^n\dot{u}^n+\partial_x P(\rho^n)-\partial_{xx}u^n=0.\end{cases} \tag{5.7}$$

5.2.2 First Hoff energy

The first estimate consists in multiplying the $(5.7)_2$ by the material derivative of u^n. The statement is the following :

Proposition 5.2.2. *There exists a constant $C(T,E_0)$ such that :*

$$\int_0^T\int_{\mathbb{T}^1}\sigma\rho^n|\dot{u}^n|^2+\frac{1}{2}\sup_{0\leqslant t\leqslant T}\sigma(t)\int_{\mathbb{T}^1}|\partial_x u^n(t)|^2\leqslant C. \tag{5.8}$$

Proof. Multiplying $(5.7)_2$ by $\dot{u}^n$ and integrating on $\mathbb{T}^1$, one obtains :

$$\int_{\mathbb{T}^1}\rho^n|\dot{u}^n|^2+\int_{\mathbb{T}^1}\dot{u}^n\partial_x P(\rho^n)+\int_{\mathbb{T}^1}\partial_x\dot{u}^n\partial_x u^n=0.$$

The last term in the above equation is :

$$\int_{\mathbb{T}^1}\partial_x\dot{u}^n\partial_x u^n=\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^1}|\partial_x u^n|^2+\frac{1}{2}\int_{\mathbb{T}^1}(\partial_x u^n)^3$$

and the second one is :

$$\int_{\mathbb{T}^1}\dot{u}^n\partial_x P(\rho^n)=-\frac{d}{dt}\int_{\mathbb{T}^1}P(\rho^n)\partial_x u^n-\int_{\mathbb{T}^1}\rho^n P'(\rho^n)\left(\partial_x u^n\right)^2.$$

Gathering the three above equations, one obtains :

$$\int_{\mathbb{T}^1}\rho^n|\dot{u}^n|^2+\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^1}|\partial_x u^n|^2=\frac{d}{dt}\int_{\mathbb{T}^1}P(\rho^n)\partial_x u^n+\int_{\mathbb{T}^1}\rho^n P'(\rho^n)\left(\partial_x u^n\right)^2-\frac{1}{2}\int_{\mathbb{T}^1}(\partial_x u^n)^3.$$

Next, one multiplies the above equation by $\sigma(s) = \min(1,s)$ and then integrates on $(0,t)$, one obtains :

$$\int_0^t \int_{\mathbb{T}^1} \sigma\rho^n|\dot{u}^n|^2 + \frac{1}{2}\sigma(t)\int_{\mathbb{T}^1}|\partial_x u^n|^2 = \frac{1}{2}\int_0^{\min(1,t)}\int_{\mathbb{T}^1}|\partial_x u^n|^2 + \sigma(t)\int_{\mathbb{T}^1} P(\rho^n)\partial_x u^n$$
$$- \int_0^{\min(1,t)}\int_{\mathbb{T}^1} P(\rho^n)\partial_x u^n + \int_0^t\int_{\mathbb{T}^1}\sigma\rho^n P'(\rho^n)\left(\partial_x u^n\right)^2 - \frac{1}{2}\int_0^t\int_{\mathbb{T}^1}\sigma\left(\partial_x u^n\right)^3. \quad (5.9)$$

The first term of the right hand side of the above equation is controlled, thanks to the classical energy estimate by $E_0/2$. As well, thanks to the boundedness of $P(\rho^n)$ and $P'(\rho^n)$ in $L^\infty((0,T)\times\mathbb{T}^1)$ (see Remark 5.2.1), for $\varepsilon > 0$, one has :

$$\sigma(t)\int_{\mathbb{T}^1} P(\rho^n)\partial_x u^n \leqslant \varepsilon\sigma(t)\int_{\mathbb{T}^1}|\partial_x u|^2 + \frac{1}{4\varepsilon}\sigma(t)\int_{\mathbb{T}^1} P(\rho^n)^2,$$
$$\int_0^{\min(1,t)}\int_0^t P(\rho^n)\partial_x u^n \leqslant \int_0^{\min(t,1)}\int_{\mathbb{T}^1}|\partial_x u^n|^2 + \int_0^{\min(t,1)}\int_{\mathbb{T}^1} P(\rho^n)^2$$

and

$$\int_0^t \sigma\rho^n P'(\rho^n)|\partial_x u^n|^2 \leqslant \int_0^t \sigma\|\rho^n P'(\rho^n)\|_{L^\infty(\mathbb{T}^1)}\int_{\mathbb{T}^1}|\partial_x u^n|^2.$$

Gathering the three above equations and owing to the boundedness of the density, one has the following inequality with a constant $C = C(\varepsilon, E_0)$

$$\sigma(t)\int_{\mathbb{T}^1} P(\rho^n)\partial_x u^n - \int_0^{\min(1,t)}\int_0^t P(\rho^n)\partial_x u^n + \int_0^t \sigma\rho^n P'(\rho^n)|\partial_x u^n|^2$$
$$\leqslant \varepsilon\sigma(t)\int_{\mathbb{T}^1}|\partial_x u|^2 + C + C\int_0^t\int_{\mathbb{T}^1}\sigma|\partial_x u^n|^2. \quad (5.10)$$

It remains estimate for the last term in (5.9).

$$\int_0^t\int_{\mathbb{T}^1}\sigma(\partial_x u^n)^3 \leqslant \int_0^t \sigma\|\partial_x u^n\|^2_{L^2(\mathbb{T}^1)}\|\partial_x u^n - P(\rho^n)\|_{L^\infty(\mathbb{T}^1)} + \int_0^t \sigma\|\partial_x u^n\|^2_{L^2(\mathbb{T}^1)}\|P(\rho^n)\|_{L^\infty(\mathbb{T}^1)}$$
$$\leqslant \frac{1}{2}\int_0^t \sigma^{1/2}\|\partial_x u^n - P(\rho^n)\|^2_{L^\infty(\mathbb{T}^1)} + \frac{1}{2}\int_0^t \sigma^{3/2}\|\partial_x u^n\|^4_{L^2(\mathbb{T}^1)}$$
$$+ \int_0^t \sigma\|\partial_x u^n\|^2_{L^2(\mathbb{T}^1)}\|P(\rho^n)\|_{L^\infty(\mathbb{T}^1)}.$$

By Gagliardo Niremberg inequality, one has :

$$\|\partial_x u^n - P(\rho^n)\|^2_{L^\infty(\mathbb{T}^1)} \leqslant 2\|\partial_x u^n - P(\rho^n)\|_{L^2(\mathbb{T}^1)}\|\rho^n\dot{u}^n\|_{L^2(\mathbb{T}^1)} \quad (5.11)$$

then

$$\frac{1}{2}\int_0^t \sigma^{1/2}\|\partial_x u^n - P(\rho^n)\|^2_{L^\infty(\mathbb{T}^1)} \leqslant \varepsilon\int_0^t\int_{\mathbb{T}^1}\sigma\rho^n|\dot{u}^n|^2 + \frac{1}{4\varepsilon}\int_0^t \|\partial_x u^n - P(\rho^n)\|^2_{L^2(\mathbb{T}^1)}$$

thus, one can write

$$\int_0^t\int_{\mathbb{T}^1}\sigma(\partial_x u^n)^3 \leqslant \varepsilon\int_0^t\int_{\mathbb{T}^1}\sigma\rho^n|\dot{u}^n|^2 + \frac{1}{4\varepsilon}\int_0^t\|\partial_x u^n - P(\rho^n)\|^2_{L^2(\mathbb{T}^1)}$$
$$+ \frac{1}{2}\int_0^t \|\partial_x u^n\|^2_{L^2(\mathbb{T}^1)}\left[\sigma\int_{\mathbb{T}^1}|\partial_x u^n|^2\right] + \int_0^t \sigma\|\partial_x u^n\|^2_{L^2(\mathbb{T}^1)}\|P(\rho^n)\|_{L^\infty(\mathbb{T}^1)}.$$

And thanks to the energy estimate, and the boundedness of the density, one has the following inequality with a constant C depending on E_0 and T :

$$\int_0^t \int_{\mathbb{T}^1} \sigma(\partial_x u^n)^3 \leqslant C + \varepsilon \int_0^t \int_{\mathbb{T}^1} \sigma \rho^n |\dot{u}^n|^2 + C \int_0^t \left[1 + \|\partial_x u^n\|^2_{L^2(\mathbb{T}^1)}\right] \left[\sigma \int_{\mathbb{T}^1} |\partial_x u^n|^2\right]. \tag{5.12}$$

Gathering (5.9), (5.10) and (5.12), taking $\varepsilon = 1/4$ one has :

$$\int_0^t \int_{\mathbb{T}^1} \sigma \rho^n |\dot{u}^n|^2 + \frac{1}{2}\sigma(t) \int_{\mathbb{T}^1} |\partial_x u^n|^2 \leqslant C + C \int_0^t \left[1 + \|\partial_x u^n\|^2_{L^2(\mathbb{T}^1)}\right] \left[\sigma \int_{\mathbb{T}^1} |\partial_x u^n|^2\right].$$

Thus applying the Grönwall lemma, and then Proposition 5.2.1 one obtains :

$$\int_0^t \int_{\mathbb{T}^1} \sigma \rho^n |\dot{u}^n|^2 + \frac{1}{2}\sigma(t) \int_{\mathbb{T}^1} |\partial_x u^n|^2 \leqslant C \int_0^t \left[1 + \|\partial_x u^n\|^2_{L^2(\mathbb{T}^1)}\right] \leqslant C'(T, E_0).$$

Which concludes the proof of the proposition. □

5.2.3 Second Hoff energy

The second Hoff energy estimate consists to apply the operator $A := \partial_t + u^n \cdot \partial_x$ to the equation $(5.9)_2$ to obtain higher order estimates. The statement is the following.

Proposition 5.2.3. *There exists a constant $C = C(T, E_0)$ such that :*

$$\sup_{0 \leqslant t \leqslant T} \sigma^2(t) \int_{\mathbb{T}^1} \rho^n |\dot{u}^n(t)|^2 + \int_0^T \int_{\mathbb{T}^1} \sigma^2 |\partial_x \dot{u}^n|^2 \leqslant C. \tag{5.13}$$

Proof. Let us begin the proof by the following computations :

$$\begin{aligned}
\int_{\mathbb{T}^1} \dot{u}^n \left(\partial_t + u^n\right)\left(\rho^n \dot{u}^n\right) &= \int_{\mathbb{T}^1} \dot{u}^n \partial_t \left(\rho^n \dot{u}^n\right) + \int_{\mathbb{T}^1} \dot{u}^n u^n \partial_x \left(\rho^n \dot{u}^n\right) \\
&= \frac{d}{dt} \int_{\mathbb{T}^1} \rho^n |\dot{u}^n|^2 - \int_{\mathbb{T}^1} \rho^n \dot{u}^n \partial_t u^n - \int_{\mathbb{T}^1} \rho^n \dot{u}^n \partial_x \left(\dot{u}^n u^n\right) \\
&= \frac{1}{2}\frac{d}{dt} \int_{\mathbb{T}^1} \rho^n |\dot{u}^n|^2 + \int_{\mathbb{T}^1} |\dot{u}^n|^2 \partial_t \rho^n - \int_{\mathbb{T}^1} \rho^n u^n \partial_x \frac{|\dot{u}^n|^2}{2} - \int_{\mathbb{T}^1} \rho^n |\dot{u}^n|^2 \partial_x u^n \\
&= \frac{1}{2}\frac{d}{dt} \int_{\mathbb{T}^1} \rho^n |\dot{u}^n|^2 + \int_{\mathbb{T}^1} |\dot{u}^n|^2 \left(\partial_t \rho^n + \partial_x \left(\rho^n u^n\right)\right) - \int_{\mathbb{T}^1} \rho^n |\dot{u}^n|^2 \partial_x u^n
\end{aligned}$$

so

$$\int_{\mathbb{T}^1} \dot{u}^n \left(\partial_t + u^n \cdot \partial_x\right)\left(\rho^n \dot{u}^n\right) = \frac{1}{2}\frac{d}{dt} \int_{\mathbb{T}^1} \rho^n |\dot{u}^n|^2 - \int_{\mathbb{T}^1} \rho^n |\dot{u}^n|^2 \partial_x u^n. \tag{5.14}$$

Next,

$$\begin{aligned}
\int_{\mathbb{T}^1} \dot{u}^n \left(\partial_t + u^n \partial_x\right) \partial_x P\left(\rho^n\right) &= \int_{\mathbb{T}^1} \dot{u}^n \partial_{tx} P\left(\rho^n\right) + \int_{\mathbb{T}^1} \dot{u}^n u^n \partial_{xx} P\left(\rho^n\right) \\
&= -\int_{\mathbb{T}^1} \partial_x \dot{u}^n P'\left(\rho^n\right) \partial_t \rho^n + \int_{\mathbb{T}^1} \dot{u}^n u^n \partial_{xx} P\left(\rho^n\right) \\
&= \int_{\mathbb{T}^1} P'\left(\rho^n\right) \partial_x \dot{u}^n \partial_x \left(\rho^n u^n\right) + \int_{\mathbb{T}^1} \dot{u}^n u^n \partial_{xx} P\left(\rho^n\right) \\
&= \int_{\mathbb{T}^1} u^n \partial_x \dot{u}^n \partial_x P\left(\rho^n\right) + \int_{\mathbb{T}^1} \rho^n P'\left(\rho^n\right) \partial_x u^n \partial_x \dot{u}^n - \int_{\mathbb{T}^1} \partial_x \left(\dot{u}^n u^n\right) \partial_x P\left(\rho^n\right) \\
&= \int_{\mathbb{T}^1} \rho^n P'\left(\rho^n\right) \partial_x u^n \partial_x \dot{u}^n - \int_{\mathbb{T}^1} \dot{u}^n \partial_x u^n \partial_x P\left(\rho^n\right).
\end{aligned}$$

By the momentum equation, one has $\partial_x P(\rho^n) = \partial_{xx} u^n - \rho^n \dot{u}^n$, then

$$\begin{aligned}\int_{\mathbb{T}^1} \dot{u}^n (\partial_t + u^n \partial_x) \partial_x P(\rho^n) &= \int_{\mathbb{T}^1} \rho^n |\dot{u}^n|^2 \partial_x u^n - \int_{\mathbb{T}^1} \dot{u}^n \partial_x u^n \partial_{xx} u^n + \int_{\mathbb{T}^1} \rho^n P'(\rho^n) \partial_x u^n \partial_x \dot{u}^n \\ &= \int_{\mathbb{T}^1} \rho^n |\dot{u}^n|^2 \partial_x u^n + \frac{1}{2} \int_{\mathbb{T}^1} |\partial_x u^n|^2 \partial_x \dot{u}^n + \int_{\mathbb{T}^1} \rho^n P'(\rho^n)\ dpxu^n \partial_x \dot{u}^n. \end{aligned} \tag{5.15}$$

As well

$$\int_{\mathbb{T}^1} \dot{u}^n (\partial_t + u^n \partial_x) \partial_{xx} u^n = -\int_{\mathbb{T}^1} \partial_x \dot{u}^n \partial_{tx} u^n - \int_{\mathbb{T}^1} \partial_{xx} u^n \partial_x (u^n \dot{u}^n). \tag{5.16}$$

The last term of the right hand side of the above equation is :

$$\begin{aligned}\int_{\mathbb{T}^1} \partial_{xx} u^n \partial_x (u^n \dot{u}^n) &= \int_{\mathbb{T}^1} \dot{u}^n \partial_{xx} u^n \partial_x u^n + \int_{\mathbb{T}^1} u^n \partial_{xx} u^n \partial_x \dot{u}^n \\ &= -\frac{1}{2} \int_{\mathbb{T}^1} |\partial_x u^n|^2 \partial_x \dot{u}^n - \int_{\mathbb{T}^1} \partial_x u^n \partial_x (u^n \partial_x \dot{u}^n) \\ &= -\frac{3}{2} \int_{\mathbb{T}^1} \partial_x \dot{u}^n |\partial_x u^n|^2 - \int_{\mathbb{T}^1} u^n \partial_x u^n \partial_{xx} \dot{u}^n; \end{aligned}$$

so

$$\int_{\mathbb{T}^1} \partial_{xx} u^n \partial_x (u^n \dot{u}^n) = \int_{\mathbb{T}^1} \partial_x (u^n \partial_x u^n) \partial_x \dot{u}^n - \frac{3}{2} \int_{\mathbb{T}^1} \partial_x \dot{u}^n |\partial_x u^n|^2. \tag{5.17}$$

Then combining (5.16) and (5.17), one has :

$$\begin{aligned}\int_{\mathbb{T}^1} \dot{u}^n (\partial_t + u^n \cdot \partial_x) \partial_{xx} u^n &= \frac{3}{2} \int_{\mathbb{T}^1} \partial_x \dot{u}^n |\partial_x u^n|^2 - \int_{\mathbb{T}^1} \partial_x (u^n \partial_x u^n) \partial_x \dot{u}^n - \int_{\mathbb{T}^1} \partial_x \dot{u}^n \partial_{tx} u^n \\ &= \frac{3}{2} \int_{\mathbb{T}^1} \partial_x \dot{u}^n |\partial_x u^n|^2 - \int_{\mathbb{T}^1} \partial_x \dot{u}^n \partial_x (\partial_t u^n + u^n \partial_x u^n) \end{aligned}$$

$$\int_{\mathbb{T}^1} \dot{u}^n (\partial_t + u^n \cdot \partial_x) \partial_{xx} u^n = \frac{3}{2} \int_{\mathbb{T}^1} \partial_x \dot{u}^n |\partial_x u^n|^2 - \int_{\mathbb{T}^1} |\partial_x \dot{u}^n|^2. \tag{5.18}$$

Applying the operator $\partial_t + u^n \cdot \partial_x$ to $(5.7)_2$, next multiplying by $\dot{u}^n$, integrating on $\mathbb{T}^1$ and gathering (5.14), (5.15) and (5.18), one obtains :

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^1} \rho^n |\dot{u}^n|^2 + \int_{\mathbb{T}^1} |\partial_x \dot{u}^n|^2 = \int_{\mathbb{T}^1} |\partial_x u^n|^2 \partial_x \dot{u}^n - \int_{\mathbb{T}^1} \rho^n P'(\rho^n) \partial_x u^n \partial_x \dot{u}^n.$$

Next multiply the above equation by σ^2 and integrate in time on $(0, t)$, one has :

$$\begin{aligned}\frac{1}{2} \sigma^2(t) \int_{\mathbb{T}^1} \rho^n |\dot{u}^n|^2 + \int_0^t \int_{\mathbb{T}^1} \sigma^2 |\partial_x \dot{u}^n|^2 &= \int_0^{\min(1,t)} \sigma \int_{\mathbb{T}^1} \rho^n |\dot{u}^n|^2 + \int_0^t \int_{\mathbb{T}^1} \sigma^2 |\partial_x u^n|^2 \partial_x \dot{u}^n \\ &\quad - \int_0^t \int_{\mathbb{T}^1} \sigma^2 \rho^n P'(\rho^n) \partial_x u^n \partial_x \dot{u}^n. \end{aligned} \tag{5.19}$$

The first term of the right hand side of the above equation is easily controlled thanks to the classical energy estimate. The last one is estimated, because of the boundedness of $P'(\rho^n)$ in $L^\infty((0,T)\times\mathbb{T}^1)$ (see Proposition 5.2.1), as follows :

$$\left|\int_0^t\int_{\mathbb{T}^1}\sigma^2\rho^n P'(\rho^n)\partial_x u^n\partial_x\dot{u}^n\right|\leqslant\varepsilon\int_0^t\int_{\mathbb{T}^1}\sigma^2|\partial_x\dot{u}^n|^2+\frac{1}{4\varepsilon}\|\rho^n P'(\rho^n)\|^2_{L^\infty(\mathbb{T}^1)}\int_0^t\int_{\mathbb{T}^1}\sigma^2|\partial_x u^n|^2.$$

Thanks to the energy estimates and the boundedness of the density, one has :

$$\left|\int_0^t\int_{\mathbb{T}^1}\sigma^2\rho^n P'(\rho^n)\partial_x u^n\partial_x\dot{u}^n\right|\leqslant\varepsilon\int_0^t\int_{\mathbb{T}^1}\sigma^2|\partial_x\dot{u}^n|^2+C_1 \tag{5.20}$$

with C_1 a constant depending only on T, ε and E_0. It remains to estimate $\displaystyle\int_0^t\int_{\mathbb{T}^1}\sigma^2|\partial_x u^n|^2\partial_x\dot{u}^n$. One has :

$$\left|\int_0^t\int_{\mathbb{T}^1}\sigma^2|\partial_x u^n|^2\partial_x\dot{u}^n\right|\leqslant\frac{1}{4}\int_0^t\int_{\mathbb{T}^1}\sigma^2|\partial_x\dot{u}^n|^2+\int_0^t\sigma^2\|\partial_x u^n\|^4_{L^4(\mathbb{T}^1)}. \tag{5.21}$$

The last term of the above equation can be estimate as follows :

$$\int_0^t\sigma^2\|\partial_x u^n\|^4_{L^4(\mathbb{T}^1)}\leqslant\int_0^t\sigma^2\|\partial_x u^n\|^2_{L^\infty(\mathbb{T}^1)}\int_{\mathbb{T}^1}|\partial_x u^n|^2\leqslant\sup_{0\leqslant s\leqslant t}\sigma^2(s)\|\partial_x u^n(s)\|^2_{L^\infty(\mathbb{T}^1)}\int_0^t\int_{\mathbb{T}^1}|\partial_x u^n|^2. \tag{5.22}$$

By the classical energy estimate, $\displaystyle\int_0^t\int_{\mathbb{T}^1}|\partial_x u^n|^2$ is controlled by E_0, but

$$\begin{aligned}\sigma^2(t)\|\partial_x u^n(t)\|^2_{L^\infty(\mathbb{T}^1)}&\leqslant 2\sigma^2(t)\|\partial_x u^n(t)-P(\rho^n(t))\|^2_{L^\infty(\mathbb{T}^1)}+2\sigma^2\|P(\rho^n)\|^2_{L^\infty(\mathbb{T}^1)}\\&\leqslant 2\sigma^2(t)\|\partial_x u^n(t)-P(\rho^n(t))\|_{L^2(\mathbb{T}^1)}\|\rho^n\dot{u}^n\|_{L^2(\mathbb{T}^1)}+2\sigma^2\|P(\rho^n)\|^2_{L^\infty(\mathbb{T}^1)}\\&\leqslant\frac{1}{2\varepsilon}\left(\sigma^2\|\partial_x u^n\|^2_{L^2(\mathbb{T}^1)}+\sigma^2\|P(\rho^n)\|^2_{L^2(\mathbb{T}^1)}\right)+\varepsilon\int_{\mathbb{T}^1}\sigma^2\rho^n|\dot{u}^n|^2\\&\quad+2\sigma^2\|P(\rho^n)\|^2_{L^\infty(\mathbb{T}^1)}.\end{aligned}$$

Owning to the previous energy estimates Proposition 5.2.2 and the boundedness of $P(\rho^n)$ in $L^\infty((0,T)\times\mathbb{T}^1)$ Proposition 5.2.1, there exists a constant $C_2(\varepsilon,T,E_0)$ such that

$$\sigma^2(t)\|\partial_x u^n(t)\|^2_{L^\infty(\mathbb{T}^1)}\leqslant\varepsilon\int_{\mathbb{T}^1}\sigma^2\rho^n|\dot{u}^n|^2+C_2. \tag{5.23}$$

Gathering (5.21), (5.22) and (5.23), one obtains :

$$\left|\int_0^t\int_{\mathbb{T}^1}\sigma^2|\partial_x u^n|^2\partial_x\dot{u}^n\right|\leqslant\frac{1}{4}\int_0^t\int_{\mathbb{T}^1}\sigma^2|\partial_x\dot{u}^n|^2+E_0\left[C_2+\varepsilon\int_{\mathbb{T}^1}\sigma^2\rho^n|\dot{u}^n|^2\right]. \tag{5.24}$$

Next, gathering (5.19), (5.20) and (5.24) one has the following estimate with a constant $C_3=C_3(T,E_0)$

$$\sigma^2(t)\int_{\mathbb{T}^1}\rho^n|\dot{u}^n|^2+\int_0^t\int_{\mathbb{T}^1}\sigma^2|\partial_x\dot{u}^n|^2\leqslant C_3\left[1+\int_0^t\int_{\mathbb{T}^1}\sigma^2\rho^n|\dot{u}^n|^2\right].$$

Applying the Grönwall lemma, one obtains :

$$\sup_{0\leqslant t\leqslant T}\sigma^2(t)\int_{\mathbb{T}^1}\rho^n|\dot{u}^n(t)|^2+\int_0^T\int_{\mathbb{T}^1}\sigma^2|\partial_x\dot{u}^n|^2\leqslant C_4(T,E_0).$$

□

5.3 Passage to the limit

According to Proposition 5.2.1 sequence $(u^n)_n$ is bounded in $L^2((0,T),H^1(\mathbb{T}^1))$, but one has only a weak convergence despite the fact that the embedding $H^2(\mathbb{T}^1) \hookrightarrow L^2(\mathbb{T}^1)$ is compact. To pass to the limit in nonlinear terms $(\rho^n u^n)_n$ and $(\rho^n (u^n)^2)_n$ one needs a strong convergence on $(\rho^n)_n$ or $(\rho^n u^n)_n$. For this end, let us use the compactness lemma of Aubin-Lions Lemma A.3.1 similar to the one of Ascoli-Arzéla.

5.3.1 Main steps of the proof

Proposition 5.3.1. *Sequence $(\rho^n)_n$ is compact in $\mathscr{C}([0,T],H^{-1}(\mathbb{T}^1))$ whereas $(\rho^n u^n)_n$ is compact in $L^2((0,T),H^{-1}(\mathbb{T}^1))$.*

Remark 5.3.1. *By this proposition, up to extraction, there exists $\rho \in \mathscr{C}([0,T],H^{-1}(\mathbb{T}^1))$ such that :*

$$(\rho^n)_n \to \rho \text{ in } \mathscr{C}([0,T],H^{-1}(\mathbb{T}^1)).$$

On the other hand, as $(u^n)_n$ is bounded in $L^2((0,T),H^1(\mathbb{T}^1))$ then, up to extraction, $(u^n)_n$ converges weakly to some u in $L^2((0,T),H^1(\mathbb{T}^1))$. Therefore $(\rho^n u^n)_n \to \rho u$ in $\mathcal{D}'((0,T)\times\mathbb{T}^1)$. This implies that $(\rho^n u^n)_n$ converges strongly to ρu in $L^2((0,T),H^{-1}(\mathbb{T}^1))$ and adding the fact that $(u^n)_n$ converges weakly to u in $L^2((0,T),H^1(\mathbb{T}^1))$ one has $(\rho^n (u^n)^2)_n \to \rho u^2$ in $\mathcal{D}'((0,T)\times\mathbb{T}^1)$. As $(\rho^n)_n$ is bounded in $L^\infty((0,T)\times\mathbb{T}^1)$, so $(\rho^n u^n)_n$ is bounded in $L^\infty((0,T),L^2(\mathbb{T}^1))$ and by the mass conservation equation, $(\partial_t \rho^n)_n$ is bounded in $L^\infty((0,T),H^{-1}(\mathbb{T}^1))$ so up to extraction, $(\partial_t\rho^n)_n \rightharpoonup^* \partial_t\rho$ in $L^\infty((0,T),H^{-1}(\mathbb{T}^1))$. Consequently, ρ and u satisfy, in sense of distribution the mass equation :

$$\partial_t u + \partial_x(\rho u) = 0.$$

For the momentum equation, $(P(\rho^n))_n$ is bounded in $L^\infty((0,T)\times\mathbb{T}^1)$ and $(u^n)_n$ is bounded in $L^2((0,T),H^1(\mathbb{T}^1))$, then $(\partial_t(\rho^n u^n) = \partial_{xx}u^n - \partial_x P(\rho^n) - \partial_x(\rho^n (u^n)^2))_n$ is bounded in $L^2((0,T),H^{-1}(\mathbb{T}^1))$ so converges weakly to $\partial_t \rho u$ in $L^2((0,T),H^{-1}(\mathbb{T}^1))$. As well $(P(\rho^n))_n \rightharpoonup^* \overline{P(\rho)}$ in $L^\infty((0,T)\times\mathbb{T}^1)$. One concludes that ρ and u satisfy the following equation in sense of distribution.

$$\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_x \overline{P(\rho)} - \partial_{xx}u = 0.$$

It remains to prove that $\overline{P(\rho)} = P(\rho)$.

We will use the renormalized solution theory introduced by Pierre Louis Lions [18], see also [21] and theory of convex function to obtain a strong convergence of $(\rho^n)_n$ to ρ in a Lebesgue space. Which ensures that $(\rho^n)_n$ converges almost everywhere to ρ and then, one can deduces that $P(\rho) = \overline{P(\rho)}$ by Lemma A.3.2.

Multiplying the mass conservation equation $(5.6)_1$ by $b'(\rho^n)$ where $b \in \mathscr{C}^1(0,1)$, one has

$$\partial_t b(\rho^n) + \partial_x(b(\rho^n)u^n) + [\rho^n b'(\rho^n) - b(\rho^n)]\,\partial_x u^n = 0.$$

In particular, for $b(t) = t\log(t)$, the above equation becomes

$$\partial_t \rho^n \log(\rho^n) + \partial_x(\rho^n \log(\rho^n) u^n) + \rho^n \partial_x u^n = 0. \tag{5.25}$$

In the following we will pass to the weak limit in the above equation. To this end, we will use Lemma A.3.3 that gives the weak limit of the product of two weak convergence in Lebesgue space.

Lemma 5.3.1. *Sequence $(\rho^n \log(\rho^n))_n$ converges weakly to some $\overline{\rho \log(\rho)}$ that satisfies we following equation :*

$$\partial_t \overline{\rho \log(\rho)} + \partial_x \left(\overline{\rho \log(\rho) u} \right) + \rho \partial_x u = \rho \overline{P(\rho)} - \overline{\rho P(\rho)} \text{ in } \mathcal{D}'((0,T) \times \mathbb{T}^1). \tag{5.26}$$

Remark 5.3.2. *Recall that $\rho \in L^\infty((0,T) \times \mathbb{T}^1)$, $u \in L^2((0,T), H^1(\mathbb{T}^1))$ satisfy*

$$\partial_t \rho + \partial_x(\rho u) = 0.$$

The function $b(t) = t\ln(t)$ is continuous on $[0,1]$ provided that it is extended by 0 at $t = 0$ and $b \in \mathscr{C}(0,1)$. As $\lim\limits_{t\to 0} t^{1/2}\log(t) = 0$ one can use b in Lemma A.3.4 and obtains :

$$\partial_t(\rho \log \rho) + \partial_x(\rho \log(\rho) u) + \rho \partial_x u = 0. \tag{5.27}$$

Then substituting (5.29) and (5.27), one has :

$$\partial_t \left(\overline{\rho \log \rho} - \rho \log \rho \right) + \partial_x \left((\overline{\rho \log \rho} - \rho \log \rho) u \right) = \rho \overline{P(\rho)} - \overline{\rho P(\rho)}.$$

In the following, we will prove that $\rho \overline{P(\rho)} - \overline{\rho P(\rho)} \leqslant 0$. To do so one will use Lemma A.3.5.

Lemma 5.3.2. *One has :*

$$\partial_t \left(\overline{\rho \log \rho} - \rho \log \rho \right) + \partial_x \left((\overline{\rho \log \rho} - \rho \log \rho) u \right) \leqslant 0. \tag{5.28}$$

The only strong continuity ρ that we have at this time is that ρ is continuous in values in $H^{-1}(\mathbb{T}^1)$. Using Lemma A.3.7, one has the following.

Lemma 5.3.3. *One has $\rho \log(\rho) \in \mathscr{C}([0,T], L^p(\mathbb{T}^1))$ for all $1 \leqslant p < 2$.*

Then integrating (5.28) on the torus, one obtains the following :

Lemma 5.3.4.

$$\int_{\mathbb{T}^1} \left(\overline{\rho \log \rho} - \rho \log \rho \right)(t) \leqslant 0$$

Therefore $\overline{\rho \log \rho} = \rho \log \rho$ almost everywhere and consequently as the function $t \mapsto t \log t$ is strictly convex, one deduces that $(\rho^n)_n \xrightarrow{n\to\infty} \rho$ strongly in $L^1((0,T) \times \mathbb{T}^1)$. This result is proved in [8]. In particular $(\rho^n)_n$ converges almost everywhere to ρ and as P is continuous and $(\rho^n)_n$ is uniformly far from 1, $(P(\rho^n))_n$ converges almost everywhere to $P(\rho)$ and as $(P(\rho^n))_n$ converges weakly to $\overline{P(\rho)}$ then $P(\rho) = \overline{P(\rho)}$. In fact this is due to Lemma A.3.2. Consequently the existence part of Theorem 5.1.1 follows.

5.3.2 Some remarks

Let us recall that we proved two additional estimates (5.8) and (5.13) that give, because of the uniform bound of ρ^n (5.5), a $L^2((0,T) \times \mathbb{T}^1)$ bound on $(\sigma^{1/2}\dot{u}^n))_n$ and the effective flux $F^n = \partial_x u^n - P(\rho^n)$ is such that $(\sigma^{1/2}\partial_x F)_n$ is bounded in $L^2((0,T) \times \mathbb{T}^1)$. This implies a weak convergence of $(\sigma^{1/2}\dot{u}^n)_n$ to some $\sigma^{1/2}v \in L^2((0,T) \times \mathbb{T}^1)$. By considering the measure σdx instead of the

Lebesgue measure dx, one proves that estimates (5.8) and (5.13) are conserved after passing to the limit. In the following lines we will prove that $v = \dot{u}$.

Coming back to (5.11), one has the following

$$\begin{aligned}\sigma^{1/2}\|\partial_x u^n\|^2_{L^\infty(\mathbb{T}^1)} &\leqslant 2\sigma\|\partial_x u^n - P(\rho^n)\|^2_{L^\infty(\mathbb{T}^1)} + 2\|P(\rho^n)\|^2_{L^\infty(\mathbb{T}^1)}\\ &\leqslant 4\sigma\|\partial_x u^n - P(\rho^n)\|_{L^2(\mathbb{T}^1)}\|\rho^n\dot{u}^n\|^2_{L^2(\mathbb{T}^1)} + 2\|P(\rho^n)\|^2_{L^\infty(\mathbb{T}^1)}\\ &\leqslant 2\sigma\|\partial_x u^n - P(\rho^n)\|^2_{L^2(\mathbb{T}^1)} + 2\int_{\mathbb{T}^1}\sigma\rho^n|\dot{u}^n|^2 + 2\|P(\rho^n)\|^2_{L^\infty(\mathbb{T}^1)}\end{aligned}$$

Then integrate in time on $(0,T)$ and use the classical energy estimate (5.3), the first Hoff energy (5.8) and the uniform bound on the density, one has :

$$\int_0^t \sigma^{1/2}\|\partial_x u^n\|^2_{L^\infty(\mathbb{T}^1)} \leqslant C_2(T, E_0).$$

Next, by Cauchy Schwartz inequality, one has :

$$\int_0^T \|\partial_x u^n\|_{L^\infty(\mathbb{T}^1)} \leqslant \left[\int_0^T \sigma^{-1/2}\right]^{1/2}\left[\int_0^T \sigma^{1/2}\|\partial_x u^n\|^2_{L^\infty(\mathbb{T}^1)}\right]^{1/2}$$

Near 0, $\sigma^{-1/2}(t)$ behaves like $t^{-1/2}$, then the factor $\int_0^T \sigma^{-1/2}$ is integral, thus one concludes that $(\partial_x u^n)_n$ is bounded in $L^1((0,T), L^\infty(\mathbb{T}^1))$. As we prove, $(u^n)_n$ converges weakly to u in $L^2((0,T), H^1(\mathbb{T}^1))$, it follows that $\partial_x u \in L^1((0,T), L^\infty(\mathbb{T}^1))$.

By the first Hoff estimate (5.8), $\left(\sigma^{1/2}\rho^{1/2}\dot{u}^n\right)_n$ is bounded in $L^2((0,T)\times\mathbb{T}^1)$ and owing to the boundedness of $(u^n)_n$ in $L^2((0,T), H^1(\mathbb{T}^1))$ and the uniform bound on the density, one has $(\sigma^{1/2}\partial_t u^n = \sigma^{1/2}\dot{u}^n - \sigma^{1/2}u^n\partial_x u^n)_n$ bounded in $L^2((0,T)\times\mathbb{T}^1)$ and so converges, up to extraction, in sense of distribution to $\sigma^{1/2}\partial_t u$. Consequently, one has u Holder in time, indeed, for almost every $0 \leqslant s < t \leqslant T$, one has :

$$u(t,x) - u(s,x) = \int_s^t \partial_t u(\tau, x)d\tau \implies \frac{\sigma^{1/2}(s)\|u(t)-u(s)\|_{L^2(\mathbb{T}^1)}}{(t-s)^{1/2}} \leqslant \|\sigma^{1/2}\partial_t u\|_{L^2((0,T)\times\mathbb{T}^1)}.$$

Furthermore, owing to the boundedness of $(u^n)_n$ in $L^2((0,T), H^1(\mathbb{T}^1))$, so $(u^n\partial_x u^n)_n$ admits a subsequence that converges in sense of distribution to $u\partial_x u$. It follows that $v = \dot{u}$. Finally, because $\partial_x u \in L^1((0,T), L^\infty(\mathbb{T}^1))$, one has :

$$u(t,x) - u(t,y) = \int_y^x \partial_x u(t,z)dz \implies \left|\frac{u(t,x)-u(t,y)}{x-y}\right| \leqslant \|\partial_x u(t)\|_{L^\infty(\mathbb{T}^1)}.$$

5.3.3 Proofs

Proof of Proposition 5.3.1. In one dimension, the Sobolev embedding $L^2(\mathbb{T}^1) \hookrightarrow H^{-1}(\mathbb{T}^1)$ is compact. As $\mathbb{T}^1$ is a bounded set, $L^\infty(\mathbb{T}^1) \hookrightarrow L^2(\mathbb{T}^1)$ is a continuous embedding, then the embedding $L^\infty(\mathbb{T}^1) \hookrightarrow H^{-1}(\mathbb{T}^1)$ is compact. From Proposition 5.2.1, one has $(\rho^n u^n)_n$ bounded in $L^2((0,T)\times\mathbb{T}^1)$ then by the mass equation $(\partial_t\rho^n)_n$ is bounded in $L^2((0,T), H^{-1}(\mathbb{T}^1))$. Then applying Lemma A.3.1 with $X_1 = L^\infty(\mathbb{T}^1)$ and $X = X_2 = H^{-1}(\mathbb{T}^1)$ one obtains the first part of the proposition.

From Proposition 5.2.1 and Remark 5.2.1, one has $(\partial_{xx}u^n)_n$ bounded in $L^2((0,T),H^{-1}(\mathbb{T}^1))$ and $(\partial_x P(\rho^n))_n$, $(\partial_x(\rho^n(u^n)^2)_n$ are bounded in $L^1((0,T),H^{-1}(\mathbb{T}^1))$, and therefore, one concludes that $(\partial_t(\rho^n u^n))_n$ is bounded in $L^1((0,T),H^{-1}(\mathbb{T}^1))$. By Hölder inequality, $(\rho^n u^n)_n$ is bounded in $L^2((0,T)\times\mathbb{T}^1)$, thus using Aubin-Lions Lemma A.3.1, one obtains the second part of the proposition. $\square$

Proof of Lemma 5.3.1. We know that $(\rho^n)_n$ is bounded in $L^\infty((0,T)\times\mathbb{T}^1)$ so, in particular, in $L^2((0,T)\times\mathbb{T}^1)$. The fact that $(\rho^n)_n$ is bounded in $L^\infty((0,T)\times\mathbb{T}^1)$ and $(u^n)_n$ is bounded in $L^2((0,T),H^1(\mathbb{T}^1))$, leads to, thanks to mass conservation equation $(5.6)_1$ $(\partial_t\rho^n)_n$ bounded in $L^2((0,T),H^{-1}(\mathbb{T}^1))$. Let us set $F^n=\partial_x u^n - P(\rho^n)$ the so-called effective flux. By the first Hoff energy estimate Proposition 5.2.2, $\left(\sigma^{1/2}\partial_x F^n\right)_n$ is bounded in $L^2((0,T)\times\mathbb{T}^1)$ and therefore

$$\lim_{\xi\to 0}\|\sigma^{1/2}\left(F^n(\cdot+\xi)-F\right)\|_{L^2((0,T)\times\mathbb{T}^1)}=0.$$

Indeed

$$|F^n(t,x+\xi)-F^n(t,x)|\leqslant\int_x^{x+\xi}|\partial_x F^n(t,y)|dy \implies$$

$$\|\sigma^{1/2}\left(F^n(\cdot+\xi)-F\right)\|_{L^2((0,T)\times\mathbb{T}^1)}\leqslant|\xi|^{1/2}\underbrace{\|\sigma^{1/2}\partial_x F^n\|_{L^2((0,T)\times\mathbb{T}^1)}}_{\text{bounded}}\xrightarrow{\xi\to 0}0$$

Then using the Lemma A.3.3 one has $\left(\sigma^{1/2}\rho^n\left(\partial_x u^n-P(\rho^n)\right)\right)_n\to\sigma^{1/2}\rho\left(\partial_x u-\overline{P(\rho)}\right)$ in $\mathcal{D}'([0,T]\times\mathbb{T}^1)$. We know that $(\rho^n P(\rho^n))_n$ is bounded in $L^\infty((0,T)\times\mathbb{T}^1)$ so, up to extraction, $(\rho^n P(\rho^n))_n\rightharpoonup^*\overline{\rho P(\rho)}$ in $L^\infty((0,T)\times\mathbb{T}^1)$ then $(\sigma^{1/2}\rho^n\partial_x u^n)_n\to\sigma^{1/2}\left(\rho\partial_x u+\overline{\rho P(\rho)}-\rho\overline{P(\rho)}\right)$ in $\mathcal{D}'([0,T]\times\mathbb{T}^1)$. Let us prove that $(\rho^n\partial_x u^n)_n\to\rho\partial_x u+\overline{\rho P(\rho)}-\rho\overline{P(\rho)}$ in $\mathcal{D}'((0,T)\times\mathbb{T}^1)$. Giving $\varphi\in\mathcal{D}((0,T)\times\mathbb{T}^1)$. There exists a large N such that supp$\varphi\subset(\frac{1}{N},T)\times\mathbb{T}^1$ so $\sigma^{-1/2}\varphi$ is well defined and with compact support but not too regular to use as test function because σ' is a Heaviside. To get around the difficulty, one use a mollifying $\sigma_\varepsilon^{-1/2}$ sequence of $\sigma^{-1/2}$. Using $\sigma_\varepsilon^{-1/2}\varphi$ as test function, one has :

$$\int_0^T\int_{\mathbb{T}^1}\sigma^{1/2}\sigma_\varepsilon^{-1/2}\varphi\rho^n\partial_x u^n\xrightarrow{n\to\infty}\int_0^T\int_{\mathbb{T}^1}\sigma^{1/2}\sigma_\varepsilon^{-1/2}\varphi\left(\rho\partial_x u+\overline{\rho P(\rho)}-\rho\overline{P(\rho)}\right).$$

By dominated convergence theorem,

$$\int_0^T\int_{\mathbb{T}^1}\varphi\rho^n\partial_x u^n=\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{T}^1}\sigma^{1/2}\sigma_\varepsilon^{-1/2}\varphi\rho^n\partial_x u^n$$

and

$$\int_0^T\int_{\mathbb{T}^1}\sigma^{1/2}\sigma_\varepsilon^{-1/2}\varphi\left(\rho\partial_x u+\overline{\rho P(\rho)}-\rho\overline{P(\rho)}\right)\xrightarrow{\varepsilon\to 0}\int_0^T\int_{\mathbb{T}^1}\varphi\left(\rho\partial_x u+\overline{\rho P(\rho)}-\rho\overline{P(\rho)}\right).$$

Consequently,

$$\int_0^T\int_{\mathbb{T}^1}\varphi\rho^n\partial_x u^n\xrightarrow{n\to\infty}\int_0^T\int_{\mathbb{T}^1}\varphi\left(\rho\partial_x u+\overline{\rho P(\rho)}-\rho\overline{P(\rho)}\right)$$

for any $\varphi\in\mathcal{D}((0,T)\times\mathbb{T}^1)$. Therefore $(\rho^n\partial_x u^n)_n\to\rho\partial_x u+\overline{\rho P(\rho)}-\rho\overline{P(\rho)}$ in $\mathcal{D}'((0,T)\times\mathbb{T}^1)$.

Also sequence $(\rho^n\log(\rho^n))_n$ is bounded in $L^\infty((0,T)\times\mathbb{T}^1)$ and by (5.25), $(\partial_t\rho^n\log(\rho^n))_n$ is bounded in $L^2((0,T),H^{-1}(\mathbb{T}^1))+L^2((0,T)\times\mathbb{T}^1)\subset L^2((0,T),H^{-1}(\mathbb{T}^1))$ then by Lemma A.3.1

$(\rho^n \log(\rho^n))_n$ is compact in $L^2((0,T),H^{-1}(\mathbb{T}^1))$, so up to extraction, sequence $(\rho^n \log(\rho^n))_n$ converges strongly in $L^2((0,T),H^{-1}(\mathbb{T}^1))$ to some $\overline{\rho\log(\rho)}$ and as $(u^n)_n \rightharpoonup u$ in $L^2((0,T),H^1(\mathbb{T}^1))$, then $(\rho^n \log(\rho^n))_n u^n \to \overline{\rho\log(\rho)}u$ in $\mathcal{D}'((0,T)\times\mathbb{T}^1)$ and $((\partial_x(\rho^n\log(\rho^n)u^n))_n \to \partial_x\left(\overline{\rho\log(\rho)}u\right)$ in $\mathcal{D}'((0,T)\times\mathbb{T}^1)$. Finally as $(\partial_t(\rho^n\log(\rho^n)))_n$ is bounded in $L^2((0,T),H^{-1}(\mathbb{T}^1))$, then up to extraction, $(\partial_t(\rho^n\log(\rho^n)))$ converges in sense of distribution to $\partial_t\overline{\rho\log(\rho)}$. Then passing to the weak limit in (5.25), one obtains :

$$\partial_t\overline{\rho\log(\rho)} + \partial_x\left(\overline{\rho\log(\rho)}u\right) + \rho\partial_x u = \rho\overline{P(\rho)} - \overline{\rho P(\rho)}. \tag{5.29}$$

□

Proof of Lemma 5.3.2. The pressure (5.1) is a non decreasing function and $(\rho^n)_n$ is bounded in $L^\infty((0,T)\times\mathbb{T}^1)$ so in $L^1((0,T)\times\mathbb{T}^1)$ then $(\rho^n)_n$ converges weakly to ρ in $L^1((0,T)\times\mathbb{T}^1)$. According to Remark 5.2.1, $(P(\rho^n))_n$ is bounded in $L^\infty((0,T)\times\mathbb{T}^1)$ so in $L^1((0,T)\times\mathbb{T}^1)$ then converges weakly to $\overline{P(\rho)}$ in $L^1((0,T)\times\mathbb{T}^1)$. It is obvious that $\Big((P(\rho^n)-\overline{P(\rho)})\rho\Big)_n \rightharpoonup 0$ in $L^1((0,T)\times\mathbb{T}^1)$ since, for any $\varphi\in L^\infty((0,T)\times\mathbb{T}^1)$ one can use $\rho\varphi$ as test function in the formulation of the weak convergence $(P(\rho^n))_n \rightharpoonup \overline{P(\rho)}$ in $L^1((0,T)\times\mathbb{T}^1)$. Also $(P(\rho^n)\rho^n)_n$ is bounded in $L^1((0,T)\times\mathbb{T}^1)$ so converges weakly to some $\overline{P(\rho)\rho}$ in $L^1((0,T)\times\mathbb{T}^1)$. Finally, for any $\varphi\in L^\infty((0,T)\times\mathbb{T}^1)$ one can use $P(\rho)\varphi$ as test function in the weak convergence $(\rho^n)_n\rightharpoonup\rho$ in $L^1((0,T)\times\mathbb{T}^1)$ and obtains $(P(\rho)\rho^n)_n \rightharpoonup P(\rho)\rho$ in $L^1((0,T)\times\mathbb{T}^1)$. Then using Lemma A.3.5, one concludes that $\overline{P(\rho)\rho}\geqslant\rho\overline{P(\rho)}$ almost everywhere and the lemma follows. □

Proof of Lemma 5.3.3. We know that $L^2(\mathbb{T}^1)$ embeds continuously (compactly) in $H^{-1}(\mathbb{T}^1)$ and the dual of $H^{-1}(\mathbb{T}^1)$, that to say $H^1(\mathbb{T}^1)$ is separable and dense in $L^2(\mathbb{T}^1)$. As by the mass conservation equation $(\partial_t\rho^n)_n$ and by (5.25), $(\partial_t(\rho^n\log(\rho^n)))_n$ are bounded in $L^2((0,T),H^{-1}(\mathbb{T}^1))$, one obtains for any $\varphi\in H^1(\mathbb{T}^1)$, mappings

$$t\mapsto\langle\rho^n(t),\varphi\rangle_{H^{-1}(\mathbb{T}^1),H^1(\mathbb{T}^1)} \ \text{ and } \ t\mapsto\langle(\rho^n\log\rho^n)(t),\varphi\rangle_{H^{-1}(\mathbb{T}^1),H^1(\mathbb{T}^1)}$$

are uniformly continuous in $t\in[0,T]$ uniformly in n. Applying the Lemma A.3.6 one obtains, $(\rho^n)_n$ and $(\rho^n\log(\rho^n))_n$ are compact in $\mathscr{C}([0,T],L^2_w(\mathbb{T}^1))$. Therefore, up to extraction

$$(\rho^n)_n\to\rho \text{ in } \mathscr{C}([0,T],L^2_w(\mathbb{T}^1)) \ \text{ and } \ (\rho^n\log\rho^n)_n\to\overline{\rho\log\rho} \text{ in } \mathscr{C}([0,T],L^2_w(\mathbb{T}^1)).$$

In particular, $\rho\in\mathscr{C}[0,T],L^2_w(\mathbb{T}^1))$ and as we know $\rho\in L^\infty((0,T)\times\mathbb{T}^1)$, $u\in L^2((0,T),H^1(\mathbb{T}^1))$ and satisfy in $\mathcal{D}'((0,T)\times\mathbb{T}^1)$ the transport equation

$$\partial_t\rho+\partial_x(\rho u)=0,$$

so by Lemma A.3.4, for all $\theta\in(0,1/2)$

$$\partial_t\rho^\theta+\partial_x(\rho^\theta u)+(\theta-1)\rho^\theta\partial_x u=0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{T}^1).$$

and together with Lemma A.3.7, one concludes that $\rho\in\mathscr{C}([0,T],L^p(\mathbb{T}^1))$ for any $1\leqslant p<2$. This last result allows us to obtain $\rho\log\rho\in\mathscr{C}([0,T],L^p(\mathbb{T}^1))$ since, by mean value theorem,

$$\|\rho\log\rho(t)-\rho\log\rho(s)\|_{L^p(\mathbb{T}^1)}\leqslant(1-\log(\alpha))\|\rho(t)-\rho(s)\|_{L^p(\mathbb{T}^1)}.$$

□

Proof of Lemma 5.3.4. Let $\varphi \in \mathscr{C}_c^1(\mathbb{T}^1)$ such that $\varphi = 1$ on $B(0,1/2)$ and supp$\varphi \subset B(0,1)$. For any $R > 0$ let us define $\varphi^R = \varphi(\cdot/R)$. One can use φ^R as test function in (5.28), and then obtains :

$$\left[\int_{\mathbb{T}^1} \left(\overline{\rho\log\rho} - \rho\log\rho\right)\varphi^R\right]_s^t - \frac{1}{R}\int_s^t\int_{\mathbb{T}}^1 \left(\overline{\rho\log\rho} - \rho\log\rho\right)u\partial_x\varphi(\cdot/R) \leqslant 0. \tag{5.30}$$

As $\rho\log\rho \in \mathscr{C}([0,T], L^1(\mathbb{T}^1))$ and $\overline{\rho\log\rho} \ \in \mathscr{C}([0,T], L^1_w(\mathbb{T}^1))$ then,

$$\overline{\rho\log\rho} - \rho\log\rho \in \mathscr{C}([0,T], L^1_w(\mathbb{T}^1))$$

so

$$\tau \mapsto \int_{\mathbb{T}^1} \left(\overline{\rho\log\rho} - \rho\log\rho\right)(\tau)\varphi^R$$

is continue, in particular

$$\lim_{s\to 0}\int_{\mathbb{T}^1} \left(\overline{\rho\log\rho} - \rho\log\rho\right)(s)\varphi^R = 0$$

Letting s goes to 0 in (5.30), one obtains, for each R,

$$\int_{\mathbb{T}^1} \left(\overline{\rho\log\rho} - \rho\log\rho\right)(t)\varphi^R - \frac{1}{R}\int_0^t\int_{\mathbb{T}}^1 \left(\overline{\rho\log\rho} - \rho\log\rho\right)u\partial_x\varphi(\cdot/R) \leqslant 0. \tag{5.31}$$

In other hand,

$$\frac{1}{R}\left|\int_s^t\int_{\mathbb{T}}^1 \left(\overline{\rho\log\rho} - \rho\log\rho\right)u\partial_x\varphi(\cdot/R)\right| \leqslant \frac{1}{R^{2/3}}\|\partial_x\varphi\|_{L^3(\mathbb{T}^1)}\int_s^t \|\overline{\rho\log\rho} - \rho\log\rho u\|_{L^{3/2}(\mathbb{T}^1)} \xrightarrow{R\to\infty} 0. \tag{5.32}$$

As well, as $\overline{\rho\log\rho} - \rho\log\rho \in \mathscr{C}([0,T], L^1_w(\mathbb{T}^1))$, $\overline{\rho\log\rho} - \rho\log\rho(t) \in L^1(\mathbb{T}^1)$. Using the fact that $\varphi^R \xrightarrow{R\to\infty} 1$ a.e and bounded in $L^\infty(\mathbb{T}^1)$, one obtains by dominated convergence

$$\int_{\mathbb{T}^1} \left(\overline{\rho\log\rho} - \rho\log\rho\right)(t)\varphi^R \xrightarrow{R\to\infty} \int_{\mathbb{T}^1} \left(\overline{\rho\log\rho} - \rho\log\rho\right)(t). \tag{5.33}$$

Finally gathering (5.31), (5.32) and (5.33) one obtains after letting R to go to infinity

$$\int_{\mathbb{T}^1} \left(\overline{\rho\log\rho} - \rho\log\rho\right)(t) \leqslant 0.$$

□

APPENDIX A

APPENDIX

In the following, we state results that we use is the note.

A.1 Liouville's transport equation

Theorem A.1.1. *Let $d \in \mathbb{N}^*$, $\Omega \subset \mathbb{R}^d$, I an interval of $\mathbb{R}^+$ and $u \colon I \times \Omega \to \mathbb{R}^d \in \mathscr{C}^1$ be vector fields. If $V \subset \Omega$ supposed to be adverted by the flow $\mathcal{X}_t$ of u. Then, for any $f \colon I \times \Omega \to \mathbb{R} \in \mathscr{C}^1$, one has :*

$$\frac{d}{dt}\int_{\mathcal{X}_t(V)} f(t,x)dx = \int_{\mathcal{X}_t(V)} \left[\partial_t f(t,x) + \operatorname{div}(fu)(t,x)\right] dx. \tag{A.1}$$

Corollary A.1.1. *Let $d \in \mathbb{N}^*$, $\Omega \subset \mathbb{R}^d$, I an interval of $\mathbb{R}^+$ and $u \colon I \times \Omega \to \mathbb{R}^d \in \mathscr{C}^1$ be vector fields of a fluid of density ρ. If $V \subset \Omega$ supposed to be adverted by the flow $\mathcal{X}_t$ of u. For $f \colon I \times \Omega \to \mathbb{R} \in \mathscr{C}^1$, let us define*

$$g(t) = \int_{\mathcal{X}_t(V)} \rho(t,x) f(t,x)dx.$$

Then

$$\frac{dg}{dt}(t) = \int_{\Omega_t} \rho(x,t)\dot{f}(x,t)dx \tag{A.2}$$

were $\dot{f} = \partial_t f + u \cdot \nabla f$ is the material derivative of f.

A.2 Grönwall's lemma

Lemma A.2.1. *Let $\varphi,\ \psi,\ y \colon [a,b] \to \mathbb{R}^+$ three positives and continuous functions satisfying :*

$$\forall\, t \in [a,b], \quad y(t) = \varphi(t) + \int_a^t \psi(s)y(s)ds \tag{A.3}$$

then

$$\forall\, t \in [a,b] \quad y(t) \leqslant \varphi(t) + \int_a^t \varphi(s)\psi(s)\exp\left(\int_s^t \psi(\tau)d\tau\right)ds.$$

Corollary A.2.1. *Let $\psi,\ y\colon [a,b] \to \mathbb{R}^+$ two continuous functions. Assume that there exists $c \geqslant 0$ such that*

$$\forall\, t \in [a,b] \quad y(t) \leqslant c + \int_a^t \psi(s)y(s)ds$$

then

$$\forall\, t \in [a,b] \quad y(t) \leqslant c \times \exp\left(\int_a^t \psi(s)ds\right).$$

A.3 Some useful results

The following lemma is the compactness lemma of Aubin-Lions similar to the Ascoli-Arzela's one.

Lemma A.3.1. *Let X_1, X and X_2 three Banach spaces. Assume that X_1 is compactly embedded X, whereas the embedding of X in X_2 just continuous. For $1 \leqslant p, q \leqslant \infty$ let us define*

$$W = \left\{u \in L^p((0,T), X_1) \colon \partial_t u \in L^q((0,T), X_2)\right\}.$$

1. *If $p < \infty$ then the embedding of W into $L^p((0,T), X)$ is compact.*
2. *if $p = +\infty$ and $q < \infty$ the the embedding of W into $\mathscr{C}([0,T], X)$ is compact.*

The following result is exercise 4.16 of [3].

Lemma A.3.2. *Let $1 < p < \infty$ and $(\varphi^n)_n$ be a sequence of $L^p((0,T) \times \mathbb{T}^1)$ such that :*

1. *$(\varphi^n)_n$ is bounded in $L^p((0,T) \times \mathbb{T}^1)$;*
2. *$(\varphi^n)_n \to \varphi$ a.e on $(0,T) \times \mathbb{T}^1$;*

Then $(\varphi^n)_n \rightharpoonup \varphi$ in $L^p((0,T) \times \mathbb{T}^1)$.

The following result is the Lemma 5.2 of [18].

Lemma A.3.3. *Let φ^n, ψ^n converge weakly to φ, ψ, respectively in $L^{p_1}((0,T), L^{p_2}(\mathbb{T}^1))$ and in $L^{q_1}((0,T), L^{q_2}(\mathbb{T}^1))$ where $1 \leqslant p_1, p_2 \leqslant \infty$,*

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = 1.$$

Assume in addition that $\partial_t \varphi^n$ is bounded in $L^1((0,T), W^{-m,1}(\mathbb{T}^1))$ for some $m \in \mathbb{N}^*$ and

$$\|\psi^n(\cdot + \xi) - \psi^n\|_{L^{q_1}((0,T), L^{q_2}(\mathbb{T}^1))} \xrightarrow{\xi \to 0} 0$$

uniformly in n. Then $\varphi^n\psi^n$ converges to $\varphi\psi$ in sense of distribution.

The following is the Lemma 6.4 of [21].

Lemma A.3.4. *Let $2 \leqslant \beta < \infty$, $b \in \mathscr{C}([0,1]) \cap \mathscr{C}((0,1]$ such that $|b'(t)| \leqslant ct^{-\lambda}$ for some positive constant c and real number λ such that $\lambda < 1$,*

$$\rho \geqslant 0 \text{ a.e and, } \rho \in L^\beta((0,T) \times \mathbb{T}^1) \text{ and } u \in L^2((0,T), H^1(\mathbb{T}^1))$$

and

$$\partial_t \rho + \partial_x(\rho u) = 0 \ \text{ in } \ \mathcal{D}'((0,T) \times \mathbb{T}^1))$$

Then,

$$\partial_t b(\rho) + \partial_x(b(\rho)u) + \left[\rho b'(\rho) - b(\rho)\right] \partial_x u = 0 \ \text{ in } \ \mathcal{D}'((0,T) \times \mathbb{T}^1)).$$

The following result is the lemma 3.35 of [21].

Lemma A.3.5. *Let Ω a bounded set of $\mathbb{R}^d$, $d \in \mathbb{N}^*$, I an interval in $\mathbb{R}$ and $P\colon I \to \mathbb{R}$ a non decreasing function. Let $(\varphi^n)_n$ a sequence of functions of $L^1(\Omega)$ with values on I such that :*

$$\begin{aligned} \varphi^n &\rightharpoonup \varphi \text{ in } L^1(\Omega);\\ P(\varphi^n) &\rightharpoonup \overline{P(\varphi)} \text{ in } L^1(\Omega);\\ P(\varphi^n)\varphi &\rightharpoonup \overline{P(\varphi)}\varphi \text{ in } L^1(\Omega);\\ P(\varphi^n)\varphi^n &\rightharpoonup \overline{P(\varphi)\varphi} \text{ in } L^1(\Omega);\\ P(\varphi)(\varphi^n - \varphi) &\rightharpoonup 0 \text{ in } L^1(\Omega). \end{aligned}$$

Then $\overline{P(\varphi)\varphi} \geqslant \overline{P(\varphi)}\varphi$.

The following result is taken from appendix C of [17].

Definiton A.3.1. *Let X be Banach space and $I \subset \mathbb{R}$ an interval. We denote by $\mathscr{C}(\overline{I}, X - w)$ the space of continuous functions on $\overline{I}$ with values in X equipped with the weak topology.*

$$\varphi\colon \overline{I} \to X \in \mathscr{C}(\overline{I}, X - w) \Longleftrightarrow \forall\, F \in X' \quad t \mapsto \langle F, \varphi(t)\rangle_{X',X} \in \mathscr{C}(\overline{I})$$

The following compactness result in $\mathscr{C}(\overline{I}, X - w)$ follows :

Lemma A.3.6. *Let X be separable Banach space and φ^n bounded in $L^\infty((0,T), X)$ for some $T > 0$. We assume that $\varphi^n \in \mathscr{C}([0,T], Y)$ where Y is a Banach space such that $X \hookrightarrow Y$, Y' is separable and dense in X'. Furthermore, we assume that for all $\psi \in Y'$, the map $t \mapsto \langle \psi, \varphi(t)\rangle_{Y',Y}$ is uniformly continuous in $t \in [0,T]$ and uniformly in n.*

Then φ^n is compact in $\mathscr{C}([0,T], X - w)$

This result is the lemma 6.15 of [21].

Lemma A.3.7. *Let $1 < \beta < \infty$, $\theta \in (0, \beta/4)$. Assume that the couple (ρ, u) satisfies :*

$$\rho \geqslant 0 \text{ a.e}, \quad \rho \in L^\beta((0,T)\times\mathbb{T}^1) \cap \mathscr{C}([0,T], L^\beta_w(\mathbb{T}^1)),\ u \in L^2((0,T), H^1(\mathbb{T}^1))$$

and satisfies

$$\partial_t \rho^\theta + \partial_x(\rho^\theta u) + (\theta - 1)\rho^\theta \partial_x u = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{T}^1).$$

Then $\rho \in \mathscr{C}([0,T], L^p(\mathbb{T}^1))$ for $1 \leqslant p < \beta$.

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